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THE PUBLISHED AND UNPUBLISHED WORK OF CHARLES STURM ON ALGEBRAIC AND DIFFERENTIAL EQUATIONS.

PRESIDENTIAL ADDRESS DELIVERED BEFORE THE AMERICAN MATHEMATICAL SOCIETY, APRIL 28, 1911.

BY PRESIDENT MAXIME BÔCHER.

Charles Sturm was born in 1803 at Geneva, then a part of France, and went to Paris at about the age of twenty-one. There he spent the rest of his life and died in 1855, having become a member of the French Academy of Sciences in 1836.

It is not necessary for us to go beyond this bare outline of Sturm's life, since it is not with his worldly fortunes that we shall be concerned. Neither do I propose to give an account of his life-work as a whole.* The brief biography and bibliography prefixed to his posthumous Cours d'Analyse fulfills to [illegible]e extent both of these purposes. We shall confine ourselves [illegible]e branch of investigation pursued by Sturm: the study of [illegible]a solutions of algebraic equations and of linear differential [illegible]tions, both ordinary and partial. It was here that Sturm's [illegible] important and suggestive work was done, and it is of [illegible]est to try to gain some insight into the relations between [illegible]arious parts of the subject as they appeared to him.

[illegible]he papers with which we are concerned may be exhibited [illegible]he following table:

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* In brief we may say that, besides the investigations with which we [illegible]ll be concerned, Sturm published

(a) An experimental memoir in collaboration with Colladon on the compressibility of liquids.

(b) A large number of minor papers, mostly geometrical.

(c) Several papers on geometrical optics including a long memoir.

(d) Some papers, partly in collaboration with Liouville, on the imaginary roots of equations, which are not without connection with Sturm's work on the real roots of algebraic equations.

XVIII 1911

THE PAPERS OF 1829, presented to the Academy on the dates given and summarised, as indicated, in the *Bulletin de Férussac.*

May 23. "Sur la résolution des équations numériques." Volume 11, pages 419–422, and volume 12, page 318, footnote.

June 1. No title given. Volume 11, pages 422–424, and volume 12, page 318, footnote. The subject of this memoir is the equation $Ax^{\alpha} + Bx^{\beta} + \cdots + Mx^{\mu} = 0$, where $\alpha, \beta, \ldots, \mu$ are real but not necessarily rational.

June 8. Note. Volume 11, page 425. It is merely stated that this note contains (1) two new proofs of the reality of the roots of the transcendental equations to which the solution of various problems in mathematical physics leads; (2) the general determination of the constant coefficients in the series for representing an arbitrary function between given limits.

July 27. "Sur l'intégration d'un système d'équations différentielles linéaires." Volume 12, pages 314–322.

August 3. "Sur la distribution de la chaleur dans un assemblage de vases." Volume 12, page 322. Nothing but the title of this paper is preserved.

October 19. "Nouvelle théorie relative à une classe de fonctions transcendantes que l'on rencontre dans la résolution des problèmes de la physique mathématique." Volume 12, page 322. Nothing but the title of paper is preserved unless, as is possible, a brief stat in volume 11, pages 424–425, refers to it.

THE THREE GREAT MEMOIRS.

1835. "Mémoire sur la résolution des équations n iques." *Mémoires des savants étrangers*, volume 6 271–318.

1836. "Mémoire sur les équations différentielles lin du second ordre." *Liouville's Journal*, volume 1, p 106–186. This memoir had been presented to Academy September 30, 1833, and an abstract o published in *l'Institut* for November 9, 1833, pages 21 223.

1836. "Mémoire sur une classe d'équations à différences partielles." *Liouville's Journal*, volume 1, pages 373–444. Cf. *l'Institut* for November 30, 1833, pages 247–248.

For the sake of completeness we note that there are also three minor papers of later date.*

It is in the three great memoirs of 1835–36 that Sturm gave ts final form to so much of his work as he completed; but the above list strongly suggests, what a closer study amply confirms, that it was in the year 1829 that the great creative period of Sturm's life fell, and that the papers presented to the Academy in that year, so far as they are still accessible, must be examined if we would gain an insight into the lines of thought followed by him in making his great discoveries. In doing this we shall find that certain not uninteresting aspects of his early work find little or no mention in the great memoirs. Most of this early work is preserved to us only in the form of brief abstracts, sometimes even only by ts title, so that some reconstruction becomes necessary. This makes it impossible for us to attain certainty at all points, but perhaps the discussion is not less interesting for this reason.

Sturm's personal and scientific relations to Fourier form an indispensable background to a consideration of the papers presented to the Academy in such rapid succession during the summer of that fruitful year. The two main subjects of Fourier's life work had been the theory of heat and the theory of the solutions of numerical equations. Both of these subjects were carried forward by Sturm, the first in the two memoirs of [illegible] the second in that of 1835. But if in the memoirs these so[illegible]ncies appear quite distinct, we find them, when we turn to to o[illegible]pers of 1829, blended in a most curious and interesting the r[illegible].

equa[illegible]ier's treatise on the solution of numerical equations was most [illegible]blished until 1831, after the author's death; but the interc[illegible]cript of this work had already in 1829 been communicated the v[illegible]eral persons among whom was Sturm, who tells us ex-
Th[illegible]y in the paper of May 23 what a strong influence it had on in t[illegible]n work.

[illegible]urier had established the theorem that a real algebraic [illegible]ation of the kth degree, $f(x) = 0$, cannot have more roots [illegible]an interval ab, neither of whose extremities is a root, than [illegible] difference between the number of variations of sign in the set of functions

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* Namely: a brief extract of a memoir written by Sturm and Liouville together: "Sur le développement des fonctions en séries . . . ," published in *Liouville's Journal*, vol. 2 (1837), pp. 220–223, and also in the *C. R.*, vol. 4, p. 675; and two papers concerning the real roots of algebraic equations in *Liouville's Journal*, vol. 7 (1842), pp. 132–133, 356–368.

$$(1) \qquad f(x),\ f'(x),\ f''(x),\ \ldots,\ f^{[k]}(x),$$

(accents denoting differentiation) at the points a and b. Sturm's theorem, as it is still called, replaces the sequence (1) by

$$(2) \qquad f_0(x),\ f_1(x),\ f_2(x),\ \ldots,$$

which coincides with (1) in the first two places, while each subsequent f_n is the negative of the remainder obtained by dividing f_{n-2} by f_{n-1},

$$(3) \qquad f_{n-2}(x) = q_{n-1}(x)f_{n-1}(x) - f_n(x) \quad (n = 2, 3, \ldots).$$

The advantage of this set over the set (1) is that the difference in the number of variations in (2) at a and b is *precisely equal* to the number of roots between a and b. Since this theorem is given in the first of the notes of 1829 and is elaborated at length in the first of the great memoirs, one might be tempted to suppose that this formed the starting point in Sturm's researches. Fortunately Sturm himself has preserved us from this mistake, for on the closing page of the first memoir of 1836 he tells us that the above theorem was merely a by-product of his extensive investigations on the subject of linear difference equations of the second order. Curiously enough, however, this subject of difference equations is nowhere else alluded to in Sturm's published writings.

The key to this difficulty lies, I feel sure, in the paper of August 3, 1829, of which as has been said only the title is preserved. This memoir is described as more extensive than the one of July 27, which, as one sees from the summary, was not brief. At Sturm's death there was found among his papers a "very extensive" memoir with almost precisely the same title,* which has also never been published. I shall try to show you how this lost paper forms the starting point in Sturm's investigations, and how all his other work which concerns us here grew directly out of it. For this purpose we must first reconstruct at least the general framework of this paper.

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* "Sur la communication de la chaleur dans une suite de vases." Cours d'Analyse, vol. 1, p. xxviii. It is there said: "Ces deux mémoires" (*i. e.*, this one and one on curves of the second order) "sont en état d'être imprimés, et M. Liouville a bien voulu se charger de leur publication." It is to be regretted that this intention was never carried out. Even at this late date the publication would be decidedly interesting if by chance the manuscript could still be found.

Suppose we have a number of vases $P_0, P_1, \ldots, P_n$ placed in any position with reference to one another and filled with various liquids at diverse temperatures. These vases we suppose to be immersed in an atmosphere which circulates freely and thus maintains a constant temperature which we take as the zero of our scale. Each of these vases radiates heat into this atmosphere, and the vases also interchange heat among themselves by radiation. Let us denote the temperature of the vase P_i at the time t by $u\,(i, t)$. The differential equation for the flow of heat is then, if we assume the Newtonian law of radiation,

$$(4)\qquad c(i)\frac{\partial u(i, t)}{\partial t} = k(i, 0)u(0, t) + \cdots + k(i, n)u(n, t)$$
$$(i = 0, 1, \ldots, n).$$

Here $c(i)$ is a positive constant depending on the specific heat of the vase P_i; when $i \neq j$, $k(i, j) = k(j, i)$ is a positive constant of proportionality which measures the amount of radiation between P_i and P_j; and finally $k(i, i)$ is written merely as an abbreviation for

$$k(i, i) = -h(i) - k(i, 0) - \cdots - k(i, i-1) - k(i, i+1) - \cdots - k(i, n),$$

where $h(i)$ is a positive constant of proportionality for measuring the radiation of P_i into the atmosphere. It is important for us to understand that the constant $k(i, j)$, when $i \neq j$, depends for its value not merely on the relative positions and the sizes and shapes of the vases P_i and P_j, but that its value is also decreased if one or more of the other vases is so placed as to cut off part of the radiation from P_i to P_j. If these vases are *completely* cut off from each other by intervening vases, the constant $k\,(i, j)$ has the value zero.

I suspect that it was precisely to the problem just indicated and in particular to the system of equations (4) that Sturm first turned his attention. There is, however, evidence (cf. the paper of July 27) that he had also under consideration problems in small vibrations and in celestial mechanics leading to systems of differential equations analogous to (4) but which may be somewhat more general in form. The paper last cited was devoted to the analytical treatment of such systems of linear homogeneous differential equations with constant coefficients. As the chief result is a method of treating the algebraic *characteristic equation* of the system, we postpone any discussion of this paper until later.

It is not, however, in more general, but rather in more special problems that Sturm found his real inspiration. Consider the case in which the vases $P_0, \ldots, P_n$ are arranged in linear sequence* and in such a way that the radiation between two non-consecutive vases is completely cut off by the intervening ones. This case is characterised analytically by all the constants $k(i, j)$ vanishing except those for which the integers i, j are either equal or differ from one another by unity. We may then write equations (4) as follows:

$$
\begin{aligned}
c(0)\frac{\partial u(0,t)}{\partial t} &= -[h(0)+k(0,1)]u(0,t)+k(0,1)u(1,t),\\
c(i)\frac{\partial u(i,t)}{\partial t} &= k(i,i-1)u(i-1,t)-[h(i)+k(i,i-1)\\
&\qquad +k(i,i+1)]u(i,t)+k(i,i+1)u(i+1,t)\\
&\qquad\qquad (i=1,2,\ldots,n-1)\\
c(n)\frac{\partial u(n,t)}{\partial t} &= k(n,n-1)u(n-1,t)\\
&\qquad -[h(n)+k(n,n-1)]u(n,t).
\end{aligned}
\tag{5}
$$

By the side of the problem in the theory of heat which we have just formulated we may advantageously consider the problem of the small transverse vibrations of a stretched elastic string whose mass is negligible but which is weighted at a number of points by heavy particles. This problem had first been considered one hundred years earlier by John Bernoulli, and for half a century this and equivalent problems had been, in more or less general forms, subjects of investigation by Daniel Bernoulli, Euler, and Lagrange;† but none of these mathematicians had gone beyond the case where the particles are of equal mass and are equally spaced,* and for fifty years the problem had practically remained untouched.

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* It is interesting to note that in the title of the manuscript found at the time of Sturm's death, and which may be supposed to be the final form which his memoir took, the phrase "une suite de vases" is used in place of the earlier "un assemblage de vases." This change suggests that as his work developed Sturm desired to give more prominence to this special case; or, indeed, he may have eliminated all consideration of the more general case.

† Cf. Burkhardt's Report in the *Jahresbericht d. deutschen Mathematiker-Vereinigung*, vol. 10 (1901–1908).

Suppose that the string in its position of equilibrium lies along the axis of x and that particles $P_0, \ldots, P_n$ with masses $c(0), \ldots, c(n)$ lie respectively at the points whose abscissas are $x_0 < x_1 < \cdots < x_n$, and let P_0 and P_n be at the ends of the string. In order to secure the same degree of generality as in the heat problem above mentioned, we assume that during the transverse vibration each particle is drawn back towards its position of equilibrium not merely by the tension of the string but also by an additional force acting towards its position of equilibrium and proportional to the distance from this point. The constant of proportionality here we denote by $h(i)$ in the case of the ith particle, and, calling the tension of the string T, we let

$$k(i, i-1) = \frac{T}{x_i - x_{i-1}}.$$

The distance of P_i from its position of equilibrium at the time t we denote by $u(i, t)$, and we assume that each particle is free to move only in a direction at right angles to the axis of x, and that the whole motion takes place in one plane. Finally we assume that the string always remains so nearly straight that the squares of the sines or tangents of the angles which its pieces make with the axis of x may be neglected.† Then it is readily seen that the equations of motion of the particles become identical with equations (5) *provided we replace the first derivatives in these equations by second derivatives.* It follows that the mechanical problem last mentioned is mathematically almost equivalent to the problem in the theory of heat considered above. That Sturm chose the latter rather than the former, with which he was surely familiar, is, so far as the greater part of his work goes,‡ a matter of slight importance and is probably to be explained by the fact that the theory of heat was at that moment a more "up to date" subject. The relation between the two problems is however so close that we may be permitted to depart so far from strict historical accuracy as to substitute the vibration problem for the heat problem in our further explanations, since in this way greater concreteness of expression may be gained.

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* In all the cases treated during the eighteenth century the particles are either supposed to be acted upon by no external forces, or to be under the influence of gravity acting in the direction of the string. The presence of such an external constant force as this is mathematically equivalent to an unequal spacing of the particles, and will therefore not be explicitly considered by us.

† It should be noticed that we do not assume the ends of the string to be fixed. If we did this, we should have a case strictly analogous to that in which the two extreme vases, P_0 and P_n, in the heat problem are maintained at the temperature zero.

‡ An essential difference occurs only in the work which leads up to the latter part of the second memoir of 1836.

We consider first the simple harmonic vibrations of the string, that is we assume that u has the form

$$u(i, t) = y(i)\,[A \sin \mu t + B \cos \mu t]. \tag{6}$$

By substituting this expression in the equations of motion we find for $y(i)$ the difference equation of the second order

$$\begin{aligned} k(i, i-1)y(i-1) - [h(i) + k(i, i-1) + k(i, i+1) \\ - \mu^2 c(i)]\, y(i) + k(i, i+1)y(i+1) = 0 \end{aligned} \tag{7}$$

or

$$\Delta\{k(i, i-1)\Delta y(i-1)\} + [\mu^2 c(i) - h(i)]y(i) = 0 \tag{7'}$$

together with the terminal conditions

$$k(1, 0)y(1) + [\mu^2 c(0) - h(0) - k(1, 0)]y(0) = 0, \tag{8}$$

$$[\mu^2 c(n) - h(n) - k(n, n-1)]y(n) + k(n, n-1)y(n-1) = 0. \tag{9}$$

The equation (7) has in general no solution other than zero which satisfies both conditions (8) and (9),—it is only for special values of μ that these conditions can both be fulfilled. Consequently we shall disregard at first condition (9), and consider merely the solution of (7) which satisfies (8). This solution obviously contains an arbitrary constant factor, since (7) and (8) are homogeneous. We therefore replace the condition (8) by the two non-homogeneous conditions

$$\begin{aligned} y(0) &= k(0, 1), \\ y(1) &= h(0) + k(0, 1) - \mu^2 c(0), \end{aligned} \tag{8'}$$

which are precisely equivalent to (8) except that (8′) also determines the otherwise arbitrary constant factor, and determines it in such a way that $y(i)$ does not vanish identically.

The solution $y(i)$ determined by (8′) does not in general satisfy

(9). It does, however, for a specified value of μ either satisfy the condition $y(n) = 0$, or a condition of precisely the form (9) where either $c(n)$ or $h(n)$ have in general been replaced by another value.* We may therefore say that for every value of μ the function $u(i, t)$ defined by (6) gives a simple harmonic vibration corresponding either to the mechanical problem we wish to consider or to a modification of this problem which consists either in having the particle P_n held fast, or in a change in the mass of this particle, or in a change in the strength of the force which pulls this particle back to its position of equilibrium. This we shall speak of as the modified problem corresponding to a given value of μ, using this term so that, for the special values of μ above referred to, the original problem itself is the modified problem.

In the plane in which the vibration takes place let us now construct an ordinate of length $y(i)$ at each of the points x_i and connect the extremities of each two successive ordinates by a straight line. The broken line thus formed, which we shall call the line $y(i)$, gives essentially the shape of the string in the simple harmonic vibration we are considering; for, if we multiply by $A \sin \mu t + B \cos \mu t$ all the ordinates of this broken line, and this will evidently not essentially affect its shape, we get a broken line which has precisely the shape of the string at the time t. The points where the line $y(i)$ meets the axis of x thus give the *nodes* of the simple harmonic vibration in question. Either from simple mechanical considerations or from the equation (7) we see that at each node the line $y(i)$ crosses the axis. Consequently, since each of the quantities $y(0)$, $y(1)$, $\cdots$, $y(n)$ is obviously a continuous function of μ (in fact a polynomial in μ^2), the nodes also vary continuously with μ, never suddenly appearing or disappearing except at the extremity x_n of the string.

There can be very little doubt that at this point Sturm, by a simple manipulation of equation (7′) which we will not stop to give here†, established the important fact, which may easily have been suggested to him by the mechanical problem itself that as μ^2 increases the abscissa of each node decreases, new nodes appearing one by one at the point x_n.

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* This is true even in the case $\mu = 0$ provided we replace $h(n)$ by a negative quantity. In all other cases positive quantities may be used in place of $c(n)$ and $h(n)$.

† Cf. Porter, *Annals of Mathematics*, second series, vol. 3 (1902), p. 55. In the article here cited Professor Porter, at my suggestion, reconstructed a part of Sturm's researches on difference equations without, however, considering either the vibration problem or the heat problem. Cf. also the article by Professor Porter's pupil, Miss Merrill, *Trans. Amer. Math. Soc.*, vol. 4 (1903), p. 423.

It is here that Sturm must have noticed the connection with Fourier's theorem concerning the roots of algebraic equations. To establish this connection we need merely to observe that the number of nodes for a given value of μ^2 (we shall write for convenience $\lambda = -\mu^2$) is simply the number of variations of sign in the set of polynomials in λ

$$(10) \qquad y(n), y(n-1), \cdots, y(0).$$

Consequently what we have said above is equivalent to saying that the number of roots of the polynomial $y(n)$ between two negative values of λ is precisely equal to the difference between the number of variations in the system (10) for these two values of λ. Sturm found, therefore, that for the particular polynomial $y(n)$ he was in possession of a sequence of polynomials of descending degrees (since $y(i)$ is a polynomial of the ith degree in λ), which served *perfectly* the purpose which the sequence of derivatives serves imperfectly in Fourier's theorem. He must then have asked himself to what properties of the polynomials (10) this fact is due, and have seen that just three properties were used:

(*a*) The last polynomial, $y(0)$, is a constant not zero.

(*b*) When one polynomial vanishes the two adjacent ones have opposite signs. This was an immediate consequence of (7), but would follow in the same way if instead of (7) the y's satisfied any difference equation

$$(11) \qquad L(i)y(i+1) + M(i)y(i) + N(i)y(i-1) = 0,$$

where L and N always have the same sign.

(*c*) The nodes increase with λ. Far less than this, however, would be sufficient for our present purpose, namely that when a node lies at x_n it decrease as λ decreases. This would in particular be the case if $y(n)$ had no multiple root and $y(n-1)$ were simply the derivative of $y(n)$, for this is precisely the property of the derivative on which Fourier's work is based.

We may then suppose that Sturm said to himself: Starting with any polynomial without multiple roots, $f(\lambda)$, in place of $y(n)$, and taking in place of $y(n-1)$ the derivative $f'(\lambda)$, How can I form a sequence of polynomials $f_2(\lambda)$, $f_3(\lambda)$, . . . , to take the place of $y(n-2)$, $y(n-3)$, . . . , which satisfy a relation of the form (11) and of which the last is a constant not zero? This question once formulated, the method of successive divisions and reversal of sign of the remainder, leading to equation (3) which is merely a special case of (11), would readily suggest itself, and Sturm's theorem in its most familiar form was found.

If our surmises so far are correct, it follows that, even at this early date Sturm must have been well aware that any sequence of functions having a small number of easily specified properties would serve the purpose of his theorem just as well as the sequence (2); so that to call such more general sequences Sturmian sequences, as is now done,* is even from a strictly historical point of view entirely suitable. Our belief that Sturm was familiar with this more general point of view need not, however, rest entirely on the line of reasoning so far explained. Not only does he show in his memoir of 1835 how other Sturmian sequences besides (2) may be formed;† but more particularly his paper of July 27, 1829, to which reference has already been made, is mainly devoted to the formation for a special algebraic equation of a Sturmian sequence which is very different from the sequence (2). If we use the notation of determinants, which Sturm does not use, the equation in question is

$$(12) \qquad \begin{vmatrix} g_{11}\lambda + k_{11} & g_{12}\lambda + k_{12} & \cdots & g_{1n}\lambda + k_{1n} \\ g_{21}\lambda + k_{21} & g_{22}\lambda + k_{22} & \cdots & g_{2n}\lambda + k_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ g_{n1}\lambda + k_{n1} & g_{n2}\lambda + k_{n2} & \cdots & g_{nn}\lambda + k_{nn} \end{vmatrix} = 0,$$

where $g_{ij} = g_{ji}$, $k_{ij} = k_{ji}$, and where the g's and k's are real and the former are the coefficients of a non-singular definite quadratic form. Sturm falls here into the error into which Laplace and Lagrange had fallen before him, and which was first corrected by Weierstrass in 1858, of thinking that this equation can have no multiple roots. He gives, however, the correct theorem that the roots are all real; and, what is important for us here, he states that the determinant in (12) and the polynomials obtained by striking off from it the last 1, 2, 3, $\cdots$ rows and columns form a Sturmian sequence,* provided each of these polynomials is multiplied by such a power of -1 as to make the coefficient of its leading term positive.†

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* Cf. Weber's Algebra, 2 ed., vol. 1, p. 303.

† Cf. also the closing lines of his abstract of May 23, 1829.

The paper of May 23, 1829, subsequently published as the memoir of 1835, did more than all of his other papers together to win for its author recognition both in France and throughout Europe. It appears to us here in its true light,‡ as a digression from his investigations in the domain of mechanics and mathematical physics. This digression was indeed carried a little farther, as the paper of June 1 testifies. Here it is shown how Fourier's methods can be applied with very slight change to obtain an upper limit for the number of real roots in a given interval for the type of transcendental equation there considered. It is clear, moreover, from a brief remark near the middle of page 424 that Sturm was here also in possession of a method of forming a Sturmian sequence.

Let us now return to the problem of the vibrating string which we were considering above. We saw that as μ^2 increases the abscissa of each node decreases. Now from (8′) and (7′) it may readily be inferred that when $\mu = 0$,

$$0 < y(1) < y(2) < \cdots < y(n),$$

so that here there is no node. On the other hand from (8′) and (7) we see that $y(i)$ is a polynomial of the ith degree in μ^2 whose leading coefficient has the sign of $(-1)^i$. Consequently, for very large values of μ^2 we have a node in each of the intervals

$$x_i < x < x_{i+1}, \qquad (i = 0, 1, \cdots, n),$$

that is, we have the maximum possible number of nodes, namely n. Accordingly as μ^2, starting with the value zero, increases, we have at first no node, then for a while one node, then for a certain interval two nodes, etc., until finally the nth node appears at x_n and from that point on we constantly have n nodes. We thus see that there are just n positive values of μ^2, for which a node lies at x_n, that is, that the equation in μ^2, $y(n) = 0$, has n distinct positive roots, and consequently, since it is of the nth degree, that it has no imaginary or negative or multiple roots. If we denote the roots arranged in order of increasing magnitude by $\mu'^2, \mu''^2, \cdots, \mu^{[n]2}$ it is clear from what has been said that for positive values of μ^2 $y(i)$ has just k nodes ($k \leqq n$) in the interval $x_0 < x < x_n$ when and only when $\mu^{[k]2} < \mu^2 < \mu^{[k+1]2}$, where, for convenience, we let $\mu^{[0]} = 0$, $\mu^{[n+1]} = + \infty$.

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* This is true only with the qualification that no two of these polynomials have a common root; cf. Weber's Algebra, 2 ed., vol. 1, p. 308. This necessary qualification is not mentioned in Sturm's abstract, though it is by no means impossible that it may have been given in the extended memoir.

† In other words, these factors are all $+1$ or alternately $+1$ and -1 according as the quadratic form of which the g's are the coefficients is positive or negative.

‡ There are further details elaborated in this memoir to which it is not necessary for us to refer.

The next step is to show that in this interval there exists one and only one value of μ^2 for which $y(i)$ satisfies the condition (9), and this follows readily from the fact* that, when $y(n) \neq 0$, $y(n-1)/y(n)$ increases as μ^2 increases, and hence increases from $-\infty$ to $+\infty$ as μ^2 increases through the interval we are now considering. This establishes the following:

Theorem of Oscillation.† There exist just $n + 1$ values of μ^2, all real and positive, for which the difference equation (7) has a solution, not identically zero, which satisfies the terminal conditions (8), (9). Denoting these values, arranged in order of increasing magnitude, by $\mu_0^2, \mu_1^2, \cdots, \mu_n^2$, the solution $y_k(i)$ of (7) corresponding to μ_k^2 and satisfying (8), (9) has exactly k nodes.

We obtain in this way $n + 1$ simple harmonic vibrations of the weighted string, which differ from one another in the number of their nodes. The most general motion of the string will be obtained by compounding these simple harmonic vibrations. Here the formulæ, which are readily obtained, for making the vibration correspond to arbitrarily given initial conditions are closely analogous to the well known ones for the representation of periodic functions at n equally spaced points by a finite trigonometric series. These latter formulæ had been obtained by Lagrange in considering a special case of the vibration problem with which we have been concerned.

In precisely the same way, the problem of the distribution of heat in a row of vases is solved by the formula

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* Cf. Porter, loc. cit. This fact is also necessary in the proof, referred to above, that the nodes decrease as μ^2 increases.

† See the articles by Porter and Miss Merrill above cited. This theorem, never published by Sturm, is the prototype of the important and still increasing class of theorems to which Klein attached this name more than fifty years later.

$$(13)\qquad A_0e^{-\lambda_0 t}y_0(i) + A_1e^{-\lambda_1 t}y_1(i) + \cdots + A_ne^{-\lambda_n t}y_n(i),$$

where $\lambda_k = -\mu_k^2$. Here also the coefficients A_k can be determined so as to correspond with given initial conditions. It was, however, not merely the analytic solution of this problem which interested Sturm, but even more, perhaps, a discussion of the properties of the solution.

If we mark the vases at any moment + or − according as their temperatures are above or below the temperature of the surrounding atmosphere, it is obvious from physical considerations that as time passes the number of variations of sign in the sequence cannot increase. This fact may be established analytically by means of the equations (5) of which the function (13) is a solution, and at the same time we can see precisely how the number of variations decreases. For a fixed value of t we plot the function (13) as a broken line, precisely as above we plotted the line $y(i)$. This line, however, unlike the line $y(i)$, does not necessarily cross the axis of x at every point where it meets it; indeed it may meet the axis of x not merely at isolated points but it may also coincide with it throughout a whole segment extending between two points x_i. Let us call each of the isolated points and segments where the broken line representing (13) meets the axis of x a node of (13). To each node we attribute a multiplicity as follows: If the node lies at, or reaches up to, one of the end-points x_0 or x_n, we take as its multiplicity the number of points x_i contained in it. Otherwise we take either this number or a number one greater, in such a way that the multiplicity shall be odd or even according as in passing through the node the function (13) does or does not change sign. This convention is justified by the fact, readily established by means of equations (5), that such multiple nodes can occur only for isolated values of t; and that for values of t a little smaller than such a value, the function (13) has exactly k simple nodes in the neighborhood of the point or segment where a k-fold node is to appear.* Now the fundamental fact here, which also follows from (5), is that as t increases through a value for which there is a node of multiplicity k, the simple nodes, after coalescing to form the multiple node, all disappear,

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* We exclude here and in what follows the possibility that the function (13) vanish identically; or, what is the same thing, we assume that not all the A's are zero.

leaving no node at all in the neighborhood of the point or segment in question, except in the one case of a node of odd multiplicity which does not lie at or reach up to one of the end-points x_0 or x_n; in which case just one simple node remains in the neighborhood in question.

From these considerations it may readily be inferred that the number of nodes of the expression

$$A_p y_p(i) + A_{p+1} y_{p+1}(i) + \cdots + A_q y_q(i), \tag{14}$$

where $A_p \neq 0$, $A_q \neq 0$, $0 \leqq p < q \leqq n$, cannot be less than p or greater than q, a multiple node being counted at pleasure either once or as often as its multiplicity indicates. For if, by introducing exponentials, we modify (14) into an expression of the form (13), the nodes of (14) appear merely as the nodes of (13) when $t = 0$, and the number of such nodes lies, by what was said above, between the number of nodes of the expression (13) in question for a very large negative, and the number for a very large positive value of t. For such extreme values, (13) coincides very nearly with constant multiples of $y_p(i)$ and $y_q(i)$ respectively.

While it does not seem likely from the scanty evidence which Sturm has left us that these latter considerations were all familiar to him in 1829, they can hardly fail to have been in his possession four years later, and it is not unlikely that if the manuscript which was among his papers at the time of his death could be recovered, it would be found to contain a systematic exposition of them along with the other matters we have touched upon.*

Two lines for further investigation now naturally presented themselves. One of these consisted in replacing the difference equation (7) and the boundary conditions (8), (9) by the more general relations of the same form

$$\begin{aligned} \Delta\{K(i)\Delta y(i)\} - G(i)y(i+1) &= 0 \quad (K(i) > 0), \\ K(0)\Delta y(0) - hy(0) &= 0 \text{ (or } y(0) = 0), \\ K(n-1)\Delta y(n-1) + Hy(n-1) &= 0 \text{ (or } y(n) = 0). \end{aligned} \tag{15}$$

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* We note in passing that if we equate the expression (13) to zero and assign to i a particular value, the equation thus obtained is precisely of the form considered in the paper of June 1, 1829 concerning which we have already spoken. Cf. the introductory remarks in the summary of that paper.

If here we assume that the quantities $K(i)$, $G(i)$, h, H are continuous increasing functions of λ, the results relating to equation (7) and also their proofs admit of ready extension to the system (15).

On the other hand we may pass over from *difference* to *differential* equations by allowing the integer n to become infinite, the points x_0 and x_n however remaining fixed. In this way the various functions of the integral argument i become functions of a continuous argument x. We thus pass from the massless string weighted by n distinct particles to the string whose mass is continuously but unequally distributed throughout its extent, and from the radiation of heat in a row of vases to the conduction of heat in a heterogeneous bar. It is this latter problem which forms the subject of Sturm's second great memoir of 1836, while the extension of the results concerning the difference equation (15) to the differential equation

$$\frac{d}{dx}\left(K(x)\frac{dy}{dx}\right) - G(x)y = 0$$

is the subject of the first memoir of that year. It is worthy of notice that in both cases Sturm used the method of passing by a limiting process from a difference to a differential equation merely as a heuristic one, making indeed hardly a mention of it in the final memoirs, and treating the differential equations *directly* by methods which are the immediate generalizations of those he had used for the difference equations in his unpublished work.* A careful examination of the abstracts published in 1829 in the *Journal de Férussac* will show that all this, at least so far as it relates to the heat problem, was in his mind even at that early date.† It was probably developed to some extent in the unpublished paper of October 19.

It is not my purpose to discuss here the two great memoirs of 1836, although the richness of their detail tempts one to linger over many points which have, it is to be feared, rather escaped the notice of mathematicians. This richness of detail probably reflects a similar quality in the earlier unpublished researches, and much of the detail there could be readily reproduced. In these days, when new methods are being suggested for obtaining a few of the fundamental results of Sturm in their simplest forms, it is not out of place to remark that if one were to cut away from Sturm's memoirs everything except what is necessary to obtain *these* results, the few pages that would be left would in brevity, rigor, and directness easily stand comparison with anything which has so far been suggested to replace them.

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* Cf. Fredholm's derivation of the theory of integral equations as a limiting case from the theory of a system of linear algebraic equations. Like Sturm, Fredholm used this limiting process merely as a heuristic one for deriving both the results and the methods for their proofs.

† There is no evidence to show whether the more general difference equation (15) was considered at this time or only at a slightly later date.

Coupled with Sturm's name in all of this work on differential equations one often finds the name of his young friend Liouville. It is true that Liouville's work on these matters was hardly inferior in originality and power to that of Sturm himself; but it must be remembered that Sturm's work was practically complete, even to the writing of the two great memoirs, before Liouville's began, and that, except for alternative proofs which the latter gave for some of Sturm's results, and for a genial extension to certain differential equations of higher order, his work dealt with a single problem, of fundamental importance it is true, which had not been treated by Sturm,* namely the proof that the development of an arbitrary function which occurs in Sturm's papers is valid. We may therefore fairly speak of the Sturm-Liouville development according to normal functions, but these normal functions themselves, and almost everything relating to their theory,† are due to Sturm alone.

I have tried to show you how all of Sturm's most important work flowed naturally from his treatment of a single physical problem, not very important in itself perhaps, certainly of no great generality or largeness of scope. Sturm's genius showed itself first in his method of handling the problem where such purely formal skill as one associates with the names of Lagrange or Poisson is less in evidence than a constant intuitional visualization of the problem combined with a sense of accuracy uncommon in his day; secondly in his perception of the relation of this problem to other questions, and to the way in which he followed up his work into adjacent fields. The power of generalizing is not rare, as the huge bulk of our current mathematical literature sadly reminds us; but one who like Sturm can seize on the important and simple modifications of a given problem has certainly one of the most essential elements of mathematical greatness.

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* One paper on this subject written in conjunction with Liouville is preserved to us in abstract (*Liouville's Journal*, vol. 2, p. 220) but was written after Liouville's first work on this subject.

† An exception should be made here of the asymptotic expressions for these functions for large values of the index. These important expressions are due to Liouville.

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A SENSUOUS REPRESENTATION OF PATHS THAT LEAD FROM THE INSIDE TO THE OUTSIDE OF AN ORDINARY SPHERE IN POINT SPACE OF FOUR DIMENSIONS WITHOUT PENETRATING THE SURFACE OF THE SPHERE.

BY PROFESSOR C. J. KEYSER.

(Read before the American Mathematical Society, April 28, 1911.)

THE logical or analytic existence of such paths—their existence in and for thought as distinguished from intuition or imagination—has been long familiar to every one, and may be made evident even to a freshman, so simple is the sufficient algebraic argument. But all efforts to envisage the paths are defeated completely.

It is the purpose of this note to show how the existence of the paths may be made evident to the intuition and even to the senses of sight and touch. The purpose is achieved by a simple transformation correlating the points of 4-space S_4 with the spheres of ordinary space S_3, including all spheres of real center and pure imaginary radius. In this way unintuitable situations in S_4, like that presented by the paths in question, are represented by intuitable analytic equivalents in S_3, and these equivalents may be rendered sensible by easily constructible physical models.

The simplest possible correlation of the kind in question is that in which the point (x, y, z, w) of S_4 and the sphere (of S_3) having (x, y, z) for center and $\sqrt{w}$ for radius shall be a pair of correspondents.

The representative in S_3 of a lineoid (an ordinary 3-space) $Ax + By + Cz + Dw + E = 0$ of S_4 is a linear complex of spheres such that, if (x_1, y_1, z_1, w_1) be a point of the lineoid, the corresponding sphere of the complex is the sphere that has its center at (x_1, y_1, z_1) and its radius equal to $\sqrt{w_1}$. Those spheres of the complex that have a same radius, $r = \sqrt{w}$, have for locus of their centers the plane $Ax + By + Cz + Dr^2 + E = 0$; and the planes that thus correspond to different values of r are parallel.

In particular, the lineoid $w = k > 0$ which is parallel to the xyz-lineoid of reference, and on the positive side of it, is represented in S_3 by the sphere complex consisting of the real spheres whose center-locus is the totality of points of S_3 and all of which have the same radius $r = \sqrt{k}$: a complex naturally to be regarded as parallel to the sphere complex composed of the points of S_3 conceived as spheres of zero radius.

The line of S_4 determined by the points (x_1, y_1, z_1, w_1), (x_2, y_2, z_2, w_2) is represented in S_3 by a range of spheres determined by the two spheres that correspond respectively to the given points. The locus of the centers of these spheres is the line determined by the points (x_1, y_1, z_1), (x_2, y_2, z_2). A sphere of the range whose center is (x, y, z), where

$$x = (x_1 + \lambda x_2)/(1 + \lambda), \quad y = (y_1 + \lambda y_2)/(1 + \lambda),$$
$$z = (z_1 + \lambda z_2)/(1 + \lambda),$$

has for radius $r = \sqrt{w}$, where $w = (w_1 + \lambda w_2)/(1 + \lambda)$; so that in general the size of a sphere of a range varies as its center moves along the locus of centers of the range, i. e., as the sphere moves along the range, generating the range as the path of a moving sphere just as in S_4 the line corresponding to the range in S_3 is the path of a moving point.

The 4-dimensional sphere $x^2 + y^2 + z^2 + w^2 - R^2 = 0$ of S_4 is represented in S_3 by a certain quadratic complex of spheres, namely, that whose spheres have for locus of centers the ensemble of points within or on the sphere $x^2 + y^2 + z^2 - R^2 = 0$ and whose radii as the distance of the center from the origin increases vary for w positive from $\sqrt{R}$ to zero and for w negative from $i\sqrt{R}$ to zero. The spheres of the complex fall into two equal classes according as the radii are real or pure imaginary.

The pair of equations

$$\begin{aligned} &x^2 + y^2 + z^2 + w^2 - R^2 = 0, \\ &w = k \quad (0 < k < R), \end{aligned} \tag{1}$$

determine an ordinary sphere (of S_4) that is immersed in the lineoid $w = k$. This sphere is represented in S_3 by a quadratic congruence of spheres, which may be called a sphered sphere. It is composed of real (visualizable) spheres whose centers have for their locus the sphere $x^2 + y^2 + z^2 = R^2 - k^2$ and whose radii are $\sqrt{k}$. This sphered sphere is immersed in the sphere complex corresponding, as above seen, to the lineoid $w = k$.

A point (x_1, y_1, z_1, w_1) of S_4 is inside, on, or outside of the sphere (1) according as $x_1^2 + y_1^2 + z_1^2 <, =,$ or $> R^2 - w_1^2$, provided $w_1 = k$. A point whose $w \neq k$ is properly neither inside, nor on nor outside the sphere (just as in ordinary space a point external to a given plane is properly neither inside, nor on, nor outside of a given circle of the plane). According as a point of S_4 is inside, on, or outside of the sphere (1), or has none of the three relations to it, so in S_3 the sphere that represents the given point is respectively inside, on, or outside the sphered sphere representing the sphere (1), or has none of the three relations to the sphered sphere.

Let (x_1, y_1, z_1, k) and (x_2, y_2, z_2, k) be two points respectively inside and outside of the sphere (1). The condition that a point

$$x = (x_1 + \lambda x_2)/(1 + \lambda), \quad y = (y_1 + \lambda y_2)/(1 + \lambda),$$

$$z = (z_1 + \lambda z_2)/(1 + \lambda), \quad w = k,$$

of the line determined by the given points shall be on the sphere (1) has a positive discriminant, showing that the line pierces the sphere in two real points. This case is the familiar one that occurs in the point geometry of S_3 and thus lies in the field of geometric intuition. In the sphere geometry of S_3 the case is represented by a sphere range determined by two spheres having $\sqrt{k}$ for radii and situated respectively inside and outside of the sphered sphere corresponding to sphere (1). It is essential to note that all the spheres of the range are of the same size and that the size is that of the spheres composing the sphered sphere, so that, as a sphere traverses the range, it twice coincides with a sphere of the sphered sphere; that is, the range as a path of a moving sphere twice penetrates the (surface of the) sphered sphere.

Now consider the lines of S_4 determined respectively by the point pairs (x_1, y_1, z_1, k), (x_3, y_3, z_3, w_3), and (x_3, y_3, z_3, w_3), (x_2, y_2, z_2, k), where $w_3 >$ or $< k$, say $w_3 > k$. No point of the former line is on sphere (1), for such a coincidence of points requires that $(k + \lambda w_3)/(1 + \lambda) = k$, a condition that is satisfied only when $\lambda = 0$, which yields the point (x_1, y_1, z_1, k), but this point is by hypothesis not on but inside sphere (1). For a like reason no point of the latter line is on sphere (1). Hence the broken line running from the first to the second and from this to the third of the given points is a path (for points) leading from the inside to the outside of sphere (1) without penetrating the surface. It is such analytically evident but unintuitable paths whose existence is to be made evident to spatial intuition and to sight and touch. The means thereto is now obvious. In S_3 the two spheres representing the first and last of the given points are respectively inside and outside of the sphered sphere representing sphere (1). The sphere representing the point (x_3, y_3, z_3, w_3) is neither inside nor on nor outside of the sphered sphere, a fact intuitively manifest because the sphere in question is larger than the spheres $(r = \sqrt{k})$ composing the sphere complex in which the sphered sphere is immersed. As to the broken sphere range representing the broken line, it is intuitively evident that it does not penetrate the sphered sphere (whose spheres are all of radius $\sqrt{k}$) for the spheres composing the first part of the broken range increase in size from that of the initial sphere (of radius $\sqrt{k}$ and lying inside of the sphered sphere) up to the final sphere (radius $\sqrt{w_3}$) of that part, whilst the spheres of the second part decrease in size from that of the sphere just mentioned down to the final sphere of the range, this sphere being outside of the sphered sphere and having $\sqrt{k}$ for radius.

To construct a model rendering the matter sensibly evident it is plainly sufficient (1) to distribute throughout a small region of space a small number of small spheres, fixing them in position as by slender vertical steel rods, some of the spheres being of equal radius r, one or more of greater radius and one or more of less, some of the spheres intersecting (though this is not indispensable) and two or more of them, if transparent, being concentric (though this feature may be omitted); (2) to construct a spherical frame (say of wire) that shall include one or more of the spheres of radius r and exclude some of them, and to locate a number of spheres of radius r so that their centers shall be points of the frame and be suitably distributed upon it; (3) to join by a rod the centers of two spheres of radius r, one of them inside the frame, the other outside, and to place

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Let the elements of the determinant be ordered according to the sum of their subscripts. In case this sum is the same for two elements these are ordered according to the value of i. Then we get an infinite sequence of elements a_{11}, a_{12}, a_{21}, a_{13}, a_{22}, ... In each infinite product of the expanded determinant order the elements which form its factors according to the same principle.

Divide a segment AB into two equal segments AC and CB, CB in two equal segments CD and DB, and so on by sequential halving of the last segment obtaining an infinite sequence of segments whose end points approach B as a limit. Repeat this construction on each of the segments AC, CD, . . . , also on each of the segments thus obtained and so on indefinitely. Consider now a term $a_{i_1j_1}a_{i_2j_2}a_{i_3j_3}\ldots$ of the expanded determinant. Let $a_{i_1j_1}$ be the k_1th element of the determinant when these are ordered as above. Now there is a perfectly definite number of elements of the determinant between $a_{i_1j_1}$ and $a_{i_2j_2}$ which may occur with $a_{i_1j_1}$ as factors in a term of the expanded determinant, that is, which are not in the same row or column with $a_{i_1j_1}$. Let this number be $k_2 - 1$. There is further a definite number of elements between $a_{i_2j_2}$ and $a_{i_3j_3}$ which may occur in a product with $a_{i_1j_1}$, $a_{i_2j_2}$. Let this number be $k_3 - 1$ and so on.

We now assign the term $a_{i_1j_1}a_{i_2j_2}a_{i_3j_3}\ldots$ to a sequence of segments on AB in the following manner: The first segment of the sequence is the k_1th segment of the sequence AC, CD, ... The second segment is the k_2th segment of the division of the segment just selected, the third is the k_3th segment of the division of the segment last selected, and so on indefinitely. We thus obtain a sequence of segments corresponding to the term $a_{i_1j_1}a_{i_2j_2}a_{i_3j_3}\ldots$, such that every segment of the sequence lies on all that precede it and also such that the lengths of the segments become indefinitely small as we proceed. Hence this sequence determines a definite point which is a common end point of all the segments of the sequence beyond a certain one or else lies within every segment of the sequence. Note that if it is a common end point of such segments then it is a left end point. In this manner we obtain on AB a point corresponding to every term of the expanded determinant. It remains to show: (1) that we thus obtain a term corresponding to every point on AB, (2) that two points thus set in correspondence to different terms of the expanded determinant are not identical.

(1) Let Q be any point on AB. In the sequence of segments

AC, CD, . . . , there is one segment within which Q lies or of which it is the left end point. (We conceive of the segment AB as extending from left to right.) Further let Q be the left end point of or lie within the k_2'th segment of the division of the k_1'th segment of AB, and so on. Thus we obtain a sequence of numbers k_1', k_2', k_3', . . . , corresponding to a definite sequence of segments which determine the point Q. The term $a_{i_1'j_1'}a_{i_2'j_2'}a_{i_3'j_3'} \ldots$ is set in correspondence with the point Q by the process described above, provided $a_{i_1'j_1'}$ is the k_1'th element of the determinant, $a_{i_2'j_2'}$ the k_2'th element after $a_{i_1'j_1'}$ of those which can occur with it in a term of the expanded determinant, and so on.

(2) Two different terms $a_{i_1j_1}a_{i_2j_2}a_{i_3j_3} \ldots$ and $a_{i_1'j_1'}a_{i_2'j_2'}a_{i_3'j_3'} \ldots$ of the expanded determinant cannot be set in correspondence to the same point by this process. Suppose their kth factors are different. For the sake of simplicity of statement we suppose the first factors $a_{i_1j_1}$ and $a_{i_1'j_1'}$ are different. Then the corresponding points P and P' are end points of or lie within different segments of the sequence AC, CD, . . . , obtained by the division of AB. Hence these points can be identical only in case one, as P, is a right end point of one of these segments and P' the left end point of the next segment. But we noted above that in case a point determined by one of these sequences is a common end point of the segments of the sequence determining it, then it must be a left end point of such segments. Hence P and P' cannot be identical.

COLUMBIA UNIVERSITY,
January 3, 1911.

SHORTER NOTICES.

The Theory of Sets of Points. By W. H. YOUNG and GRACE CHISHOLM YOUNG. Cambridge, University Press, 1906. xii + 316 pp.

THIS volume consists of a systematic presentation of the theory of sets of points. The titles of the chapters are: Rational and irrational numbers, Representation of numbers on the straight line, The descriptive theory of linear sets of points, Potency and the general idea of a cardinal number, Content, Order, Cantor's numbers, Preliminary notions of plane sets, Regions and sets of regions, Curves, Potency of plane sets, Plane content and area, Length and linear content.

While the main body of the subject matter is thus seen to consist of those well-known elements of point set theory developed by Cantor and others, there is nevertheless considerable material which is original with the authors. As stated in the preface: "The writing of the book has necessarily involved attempts to extend the boundaries of existing knowledge and to fill in gaps which broke the connection between isolated parts of the subject."

Chapter I on rational and irrational numbers contains only eight pages and consequently a good deal is here necessarily left to the reader. The chapter ends with the theorem of Liouville.

In Chapter II a correspondence preserving order is set up by means of projection between the numbers of the real number system and the points of a line. This enables the author, without employing the notion of measurement, to use interchangeably the points on the line and the real numbers in discussions involving order and the number of points (potency) of a set. This correspondence cannot however furnish a basis for the treatment of content of sets of points since the relative lengths of segments on a line would depend upon the choice of the points 0, 1, and ∞. That is, it would depend upon the point at which one starts measuring and also upon the unit of measure. Clearly the property of relative lengths of segments on a line must be a property of those segments themselves and cannot depend on any arbitrary choice. In Chapter V, where content is considered, no mention is made of this fact and the unwary reader might be led to believe that this projective correspondence is all that is needed. Theorems relating to order, limiting points, character of sets as to density and potency are all valid without a basis in stronger congruence assumptions but, in general, theorems comparing the lengths of segments are not.

Chapter III deals mainly with what are called derived and deduced sets. A derived set is the set consisting of all the limit points of a set. If A_1 is a derived set of A and A_2 a derived set of A_1 and so on indefinitely then the set common to all the sets $A_1, A_2, \cdots$ is the deduced set of this set of sets of points. In general the process of taking all points common to an infinite series of sets of points is called deduction. An example is given of sets which have an infinite series of derived sets, all different, a first deduced set which in turn has an infinite series of derived sets all different, this series a deduced set and so on indefinitely. This serves as a concrete example in which the transfinite numbers of Cantor appear naturally and is later (Chapter VIII) referred to in connection with the formal treatment of these numbers. This is one instance of what seems one of the noteworthy features of this book, viz., the large number of simple and highly instructive examples.

Chapter IV. treats of potency and the generalized idea of cardinal numbers. Several theorems due to Mr. W. H. Young are here given. Among these are the theorems on overlapping intervals covering any set and the generalized Heine-Borel theorem.*

We note the following definitions: "If $G_1, G_2, \cdots$ be a series of sets of points such that, for all values of n, G_n contains G_{n+1} and if G be the set of all the points common to the sets G_n, G is called the *inner limiting* set of the series. $\cdots$" "If in the preceding G_n is contained in G_{n+1} and G be the set such that every set G_n is contained in G, while every point of G belongs to some definite G_n, G is said to be the *outer limiting* set of the series." Surely this terminology is much to be preferred to the less natural expressions greatest common divisor and lowest common multiple used by Cantor and his continental followers. The most important innovation in this chapter is the treatment of derived and deduced sets (adherences and coherences) without the use of Cantor's numbers. The doubt expressed in the *Fortschritte der Mathematik*, page 530, volume 34, as to the possibility of treating this subject without the use of Cantor's numbers cannot possibly have any foundation. Explicitly the question is as to whether a theory which is a special case of a more general theory may not be discussed on its own merits without reference to that more general theory. Stated in this form the question is trivial. In the case of a difficult subject like the transfinite numbers in their full generality it would appear good procedure to study a less general case of the subject first. In so far as this more special case involves properties that are taken account of in the case of the more general treatment of transfinite numbers, in so far the discussion must involve notions contained in the more general treatment. The remark by Schoenflies "H. W. Young benützt bei seinen Beweisen, die sachlich darauf hinauslaufen, eine wirkliche sukzessive

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* The latter was also given by H. Lebesgue in his thesis, Paris, 1902, the same year that Young published the theorem in the *Proc. L. M. S.*

Analyse der Punktmengen vorzunehmen, nicht ausdrücklich aber doch stillschweigend die Cantorschen Zahlen der zweiten Klasse $\cdots$"* is therefore also trivial.

The definitions of content, Chapter V, are novel. "The content I_p of a closed set of points is the difference between the content of the fundamental segment and that of those intervals of the fundamental segment which contain no points of the set." (It is proved earlier that every closed set may be obtained as the remainder of a segment when a certain set of open segments, possibly an infinite set, has been removed.) The *inner content* of any set is defined as the upper limit of the contents of its closed components. The *outer content* is the lower limit of the content of all sets of intervals covering the set. The outer content or measure of Young is identical with that of Lebesgue, while the inner measure of Lebesgue is obtained by subtracting the exterior measure of the complementary set from the whole segment. It is rather difficult to decide which of these two definitions serves the purpose in hand more directly. Schoenflies with his usual air of finality remarks: "Statt Lebesgues natürlicher Definitionen stellt H. W. Young die folgenden künstlichen an die Spitze."† It is then shown that the question as to whether or not all sets are measurable (have identical inner and outer content) depends upon whether or not there exist two open sets with no point in common, each of inner content zero, whose sum is a closed set of content not zero.

The following theorem due to W. H. Young is worthy of note: "If $G_1, G_2, \cdots, G_n, \cdots$ is a sequence of sets of points, each of which sets is a component of a closed set of finite content l, and if the interior measure of each of the sets $G_1, G_2, \cdots, G_n, \cdots$ is greater than a fixed number e, then there exists a set of points of interior measure $\geqq e$, and of the power of the continuum, such that each point of the set belongs to an infinite number of the given sets." This is regarded by the authors as one of the most important in the whole theory of linear content.

The subject of content is completed in Chapter XII, where the content and area of plane regions are discussed in full detail and the way is pointed to a similar discussion for the n-dimensional region.

Chapters VI and VII are devoted to order and the Cantor numbers respectively. Particular care is taken to define order with respect to the set of points whose order is being defined and not with respect to the fundamental set, e. g., the continuum.

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* Bericht über Punktmannigfaltigkeiten, zweiter Teil, page 76.

† Loc. cit., p. 88.

In Chapter VIII a brief resumé is given of the kinds of (1, 1)-correspondence possible between the points of a planar and a linear continuum, "space filling" curves, etc. The following statement is evidently meant to indicate the authors' point of view in regard to the "crinkly" curves but does not seem very definite: "··· the concept of a curve, as such, regarded as a set of points, must surely be recognized as a conglomeration of ideas of which that of order is only one, and is from many points of view a subservient one. A curve like any other plane set of points, has a *form*, (the italics are the reviewer's) and it is in many respects this form which is its most interesting characteristic" (page 168). This might conceivably be a very interesting statement if we knew what definite concept to attach to the word "form."

The notion of *region* is developed from the simplest region, viz., the triangular region. A region is namely a part of a plane which may be tiled over by a set of triangles (finite or infinite). A region may or may not contain rim points (points which are limit points of points not in the region) while a *domain* is definitely defined to be a region which does not contain a single rim point. This definition of domain is the basis for one of the several inaccurate statements which to a certain extent mar this chapter.

The statement "The part of a domain left over after removing a domain contained in the first (difference of two domains) is a domain or domains" (page 196) is obviously not true. Again Theorem 10, page 193, should read: "If the regions have *at least* two common points, and also any further common points lie on the same straight line as those two, then (1) the span in one direction (viz., that perpendicular to the line of common points) decreases without limit; (2) the spans in any other direction have a positive lower bound."

The corollary under Theorem 5, page 187, is not true. Besides the set specified in the corollary there may be a set of isolated points or a discrete set with limit points. Page 187, line 26, read "spans" for "span" and "limit" for "limits." In the definition of disc, page 188, insert after "meets it" the words "in an internal point." The definition of connected set leads to curious results: "A set of points such that, describing a region in any manner round each point and *each limiting point* (the italics are the reviewer's) of the set as internal point, these regions always generate a single region, is said to be a connected set, provided it contains more than one point. Hence if a set is connected, the set got by closing it is connected and vice versa" (page 204). This definition, which is shown to be equivalent to that given by Cantor in purely metric terms, makes the inside of triangle connected with its outside, and in general two sets neither of which contains a limit point of the other may form one connected set.

The following statement requires lenient construction to make it true: "Given any set of regions, a (countable) set of non-overlapping regions is uniquely determined, having the same internal points as the given set" (page 199). What meaning can be given to the words "uniquely determined" to make this hold?

These various errors are apparently due to temporary inattention rather than to lack of insight. In no case do they lead to grave results.

A curve is defined as follows: "A plane set of points, dense nowhere in the plane, such that, given any norm e, and describing around each point of the set a region of span less than e, these regions generate a single region R_e, whose span does not decrease indefinitely with e, is called a curved arc, or shortly a curve" (page 219). This definition is essentially new, differing radically from that of Schoenflies who regards a curve as the frontier of a region, or from that of Veblen which is given in terms of order and linear continuity.* As will readily be noticed, this curve is very different from an arc of a Jordan curve in that it may consist of a network of such curves as complicated as we please, —indeed of infinite number of "branches" of such curves. In dealing with the separation of the plane by polygons and curves a great deal is left to the reader. The general outline of a fuller treatment may be said to be given but it would be no mean task to fill it out in detail.

Chapters XII and XIII, dealing with plane content and area, and length and linear content, make such use of the chapters that precede as is logically necessary, and hence the character of these last chapters of the book is practically determined by the earlier parts. Chapters IX, X, and XI are not required for the reading of these last chapters and that on plane content.

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* *Transactions Amer. Math. Society*, vol. 6 (1905), pp. 83–98.

The figures employed are all of an elementary nature, circles, rectangles, triangles, and squares. On the other hand some of the results of these chapters are stated in the book for the first time. Schoenflies (Bericht, II, page 93) has incorrectly reported the decisive examples on pages 275–283. The surprising result that a countable closed set of points in the plane may have positive linear content, which may even be infinite, has entirely escaped comment in the "Bericht." An appendix deals with various questions that arose during the printing of the book but too late for insertion in the main body of it. At the end a very full bibliography is given,—so full as to be rather discouraging. It would be of real service if the book contained in a short compass a statement of the contributions to point set theory by the different investigators.

Throughout there are many well constructed figures which assist the reader very materially. The large number of problems exhibiting a multitude of phases of the subject bear eloquent witness to the care with which the authors themselves have mastered the subject and the great amount of energy expended in writing the book.

Inasmuch as this is practically the only treatise of its kind (Schoenflies's Bericht apparently having a quite different purpose), it is difficult to judge how greatly it differs from the riper treatises which are bound to come in the future. But we are surely justified in saying that the authors have done the cause of mathematics a real service by placing at the disposal of the student a treatment of point sets exceptionally readable and with unimportant exceptions entirely trustworthy.

A very considerable number of misprints have been detected.

N. J. LENNES.

Leçons élémentaires sur la Théorie des Fonctions analytiques. 2nd edition. By ÉDOUARD A. FOUËT. Vol. I: *Les Fonctions en général.* 1907. xiii + 112 pp. Vol. II: *Les Fonctions algébriques. Les Séries simples et multiples. — Les Intégrales.* Paris, Gauthier-Villars, 1910. xi + 263 pp.

A COMPARISON of the first and second editions shows that the author has completely revised his work, adding new material to include some of the latest developments, enlarging some of the subjects already treated and rewriting other portions. The one volume of the first edition has been expanded into two.

In the first volume the author seeks to give a general introduction to analytic functions, laying special emphasis upon the fundamental ideas, and indicating the growth of these ideas from the simplest naive notions to the present extended conceptions. Thus, the growth of the idea of function is traced from the earliest analysts, who restricted the use of the word to the powers of a quantity, through the various stages represented by Euler, Cauchy, Riemann, Fourier, and Dirichlet, to the present general conception which is broad enough to include a correspondence which represents a space-filling curve.

After treating the subjects of number and function, the author takes up at some length the principles of assemblages. He then considers the principal types of functions and their classification. He introduces the subject of the general analytic function, gives the Cauchy-Riemann partial differential equations, studies conformal representation, considers various singularities, introduces groups and applies them to periodic and automorphic functions.

The second volume is a study of functions from the three points of view suggested by the names of Riemann, Weierstrass, and Cauchy. The first chapter, on algebraic functions, is dominated by the geometric point of view. It studies various transformations, the fundamental properties of algebraic functions, and contains an exposition of the Riemann surface.

The subject of series is treated very fully. A general discussion of convergence is followed by a treatment of the power series, infinite products, trigonometric series, and divergent series (presenting largely the points of view of Poincaré and Borel). The theory is followed by applications to e^z, sin z, $\zeta(z)$ of Riemann, etc., and by a consideration of the hypergeometric series.

Series in two or more variables are also studied, and applied to various transcendental functions, e. g., the ζ, γ, γ' functions of Weierstrass, the θ functions of Jacobi, etc.

The last chapter deals with functions as defined by the definite integral, the epoch-making conception of Cauchy, which, according to Hermite, is the most fruitful conception in mathematics. The objective of this chapter is the integrals of Cauchy and Taylor's series.

Throughout the two volumes we find a double purpose: the author seeks to give an elementary account of the subject from the most modern standpoint; and aims, at the same time, at completeness. He accomplishes his purpose very well by the introduction of copious footnotes which deal with sources, historical information, difficult demonstrations, interesting generalizations, and conceptions too advanced for the text proper. Thus, the discussion of the theory of assemblages is supplemented in the footnotes by reference to transfinite assemblages; and the ordinary presentation of the integral is supplemented by the introduction of upper and lower integrals.

The comprehensive, suggestive, critical footnotes greatly enhance the value of the work.

The volumes under review are especially valuable for those who wish a thorough treatment of the fundamental conceptions, and an introduction to the latest ideas, for the author has not only given a sketch of these phases of the subject, but has also indicated many sources.

GEO. N. BAUER.

Lectures de Mécanique: La mécanique enseignée par les Auteurs originaux. Par E. JOUGUET. Deuxième partie: *L'Organisation de la Mécanique.* Paris, Gauthier-Villars, 1909. 284 pp.

MANY of the mathematicians who teach the elements of mechanics may have more or less serious *arrières pensées* relative to the way in which they have presented the fundamental concepts to their students, and they may form many a good resolution as to the severe logical thinking they will expend upon the subject to the end that the next time they teach it the presentation may gain much in completeness and consistency. To all such Jouguet's Lectures are a godsend,—not that all the difficulties of the doubting ones will be relieved by the perusal of the work, but that the doubts and perplexities of the great creators of mechanics, and the way they settled them or at least thrust them aside, are here detailed. For the plan of the work, as the subtitle indicates, is to teach (the foundations of) mechanics by (large extracts from) the original authors. We may say that Jouguet selects his quotations well and makes each one sufficiently long to be intelligible of itself; but one must add that his own careful critical comments are very helpful toward the fullest interpretation both of the material cited and of the subject itself.

The work consists of three parts: the first, which is not at hand, called La naissance de la mécanique; the second, which is under review, entitled L'organisation de la mécanique; the third, which as yet is only projected, to be designated Les bornes de la mécanique. Although the first and last parts must of necessity throw much light upon the second, it should be observed that this second, which has chiefly to do with the dynamics of a particle and rigid body and which as such bears most closely on the problems of interest to the teacher, may without difficulty be read alone. And the reading is highly to be recommended. The extremely logical may long for a specific set of postulates by virtue of which the subject matter and terms of elementary mechanics shall be defined; but though the actual establishment of such a set be a worthy scientific objective point, one should recognize that it may have much of the sterility of any puristic accomplishment,—the instructive thing is to contemplate the foundations of a subject from various angles, and this is precisely what Jouguet makes easy for us in reference to mechanics.

We have written from the point of view of the teacher, but the student, the physicist, the philosopher will find much of interest to them in these pages, and the logician who would construct a set of postulates for mechanics which should be of real value to teacher and pupil as well as of scientific exactness cannot safely neglect the points of approach which the race, through its geniuses, has shown to be the genius of the race.

E. B. WILSON.

CORRECTION.

PROFESSOR D. R. Curtiss calls our attention to the following erratum in his paper published in the BULLETIN in June last:

Page 466. In the formula for the determinant $\triangle_n(x, \xi)$, the index $n-2$ should, throughout the last row, be replaced by $n-3$.

NOTES.

THE concluding (July) number of volume 12 of the *Annals of Mathematics* contains the following papers: "The harmonics of a stretched string vibrating in a resisting medium," by C. R. DINES; "The mixing effect of surface waves," by C. S. SLICHTER; "Characteristics of two partial differential equations of order one," by C. A. NOBLE; "The differential equation of the third order with a quadratic equation between the integrals," by S. EPSTEEN; "Relations among some cyclotomic numbers," by T. HAYASHI.

THE eighty-first annual meeting of the British association for the advancement of science was held at Portsmouth, England, August 30 to September 6, under the presidency of Sir William Ramsey. Professor H. H. TURNER was chairman of section A, mathematics and physics.

THE fortieth annual meeting of the French association was held at Dijon, July 31 to August 5, with Inspector Ch. Lallemand as president. E. BELOT was chairman of the mathematical section.

THE fifth annual meeting of the Italian association will be held at Rome, October 12 to 18; Professor G. Ciamician is president, and Professor G. CASTELNUOVO chairman of section A.

THE second congress of Scandinavian mathematicians was held at Copenhagen, August 28–31.

THE organizing committee of the Fifth international congress of mathematicians, to be held at Cambridge, England, August 22–28, 1912, has issued a circular containing the names of the 46 members of the international committee and giving the following items of information:

"Special arrangements will be made for the consideration of the reports of the international committee appointed at the congress in Rome to enquire into the teaching of mathematics.

The committee has commenced to organize a series of lectures which shall give some idea of the present state and progress of the principal branches of mathematics, including its applications. The speakers already chosen are E. BOREL, E. W. BROWN, A. KNESER, E. G. H. LANDAU, Sir J. LARMOR, and Sir W. WHITE.

Their subjects will be announced later. The congress will be divided into four sections: I, arithmetic, algebra, analysis; II, geometry; III, applications; IV, philosophical questions."

AMONG the prizes awarded by the Paris academy of sciences at the close of the second semester, the following were for achievements in pure and applied mathematics: The Lalande prize (fr. 540) to Dr. Lewis Boss, of the Albany Observatory; the Pontécoulant prize (fr. 700) to Dr. L. SCHULOF, for his theory of comets and the completion of the lunar tables; the Francoeur prize (fr. 1000) to E. LEMOINE for his work in geometry; the Poncelet prize (fr. 2000) to M. RATEAU for progress in mechanics; the Montyon prize (fr. 700) to M. JOUGUET, for his work in thermodynamics. A general prize of fr. 10,000 was awarded to the estate of the late Professor J. TANNERY for his work on the theory of functions and particularly for his services as secretary to the Ecole Normale.

THE section of physics and mathematics of the royal society of Naples announces the following prize problem:

"A new contribution to the theory of differential forms of general order and degree."

Competing memoirs should be written in Italian, French or Latin, and sent to the secretary of the academy before June 30, 1912. The prize is 500 lire.

FROM the Dr. Elsa Neumann foundation the University of Berlin will award a prize of 1000 marks to the author of the most meritorious contribution in mathematics or physics submitted to the philosophical faculty of the university during the year 1911. The award will be made February 18, 1912.

THE Provisional Report of the national committee of fifteen on a geometry syllabus, has now been issued in pamphlet form. It is a reprint of the partial reports published in *School Science and Mathematics*, during April, May, and June, 1911, and was submitted in its present form to the National educational association at its San Francisco meeting, July 8–14, 1911. The report contains a historical introduction of 25 pages prepared by Professor F. Cajori; a section on logical considerations including axioms, definitions, treatment of limits, time and place in the curriculum, purpose of the study of geometry; a section on the grading and distribution of exercises; a section on types of courses for special classes of students and on preliminary inductive courses in the grades; and finally the syllabus itself, exhibiting by means of different forms of type the varying degrees of emphasis which may properly be attached to the various theorems.

At the meeting of the International commission on the teaching of mathematics held at Milan, Italy, September 18–21, 1911, the central committee received reports of the various national sub-committees and formulated plans regarding the submission of the final report to the Fifth international congress of mathematicians to be held at Cambridge, England, next August. From the reports of the various sub-committees the following questions were discussed at length:

A. To what extent can the systematic presentation of mathematics in the advanced schools (colleges, lycées, Gymnasien, Realschulen, $\cdots$) be taken account of? Is it wise to emphasize the fusion of different branches in such instruction?

B. What is the proper instruction in mathematics both pure and practical, for students of physics and the natural sciences?

The annual list of American doctorates published in *Science* presents for the academic year 1910–1911, 437 names, of which 239 are credited to the sciences. The following 26 successful candidates offered mathematics as major subject (the titles of the theses are appended): H. L. Agard, Yale, "The extension of some theorems in the theory of sets of points in n-dimensional space"; T. B. Ashcraft, Johns Hopkins, "Quadratic involutions on the plane rational quartic"; Miss C. L. Bacon, Johns Hopkins, "The Cartesian oval and the elliptic functions"; R. P. Baker, Chicago, "The problem of the angle bisectors"; Miss I. Barney, Yale, "Line and surface integrals"; W. H. Bates, Chicago, "An application of symbolic methods to the treatment of mean curvature in hyperspace"; F. W. Beal, Princeton, "Associated normal congruences"; Miss A. D. Biddle, California, "Constructive theory of the unicursal plane quartic by synthetic methods"; P. P. Boyd, Cornell, "On the perspective Jonquières involutions associated with the (2, 1) ternary correspondence"; D. Buchanan, Chicago, "A class of periodic solutions of the problem of three bodies, two of equal mass, the third moving on a straight line"; B. H. Camp, Yale, "The convergence of singular integrals"; R. D. Carmichael, Princeton, "Linear difference equations and their analytic solutions"; J. L. JONES, Yale, "Number concept"; S. LEFSCHETZ, Clark, "On the existence of loci with given singularities"; L. LINDSAY, Syracuse, "The minors of a compound determinant"; W. R. MARRIOTT, Pennsylvania, "The determination of the order of the groups of isomorphisms of the groups of order p^4, where p is a prime"; W. O. MENDENHALL, Michigan, "On the characteristic properties of sum formulas in the theory of divergent series"; W. J. MONTGOMERY, Clark, "Singularities of twisted quintic curves"; L. O'SHAUGHNESSY, Pennsylvania, "The integrability of the differential equation representing the sum of a family of series"; A. D. PITCHER, Chicago, "The interrelations of eight fundamental properties of classes of functions"; H. W. REDDICK, Columbia, "Systems of tautochrones in a general field of force"; E. B. STOUFFER, Illinois, "Invariants of linear differential equations with applications to projective differential geometry"; S. E. URNER, Harvard, "Certain singularities of point transformations in space of three dimensions"; Miss W. P. WEBBER, Cincinnati, "On the construction of doubly periodic functions which have singular points (polar and essential) in the period parallelogram"; Miss M. B. WHITE, Chicago, "The dependence of the focal point on curvature in space problems of the calculus of variations"; W. A. WILSON, Yale, "Theory of point aggregates applied to Lebesgue integrals."

HARVARD UNIVERSITY.—During the academic year 1911–12 Professor G. A. Bliss of the University of Chicago will spend the months October to December, and Professor Max Mason of the University of Wisconsin the months February to June as Lecturers in Mathematics at Harvard. Besides taking part in more elementary instruction, they will give the following advanced courses: Professor BLISS: Differential geometry of curves and surfaces, three hours; Partial differential equations, three hours. Professor MASON: Dynamics of rigid and elastic bodies, three hours; The electron and the electro-magnetic field, three hours.

The following advanced courses are also announced:

By Professor W. E. BYERLY: Introduction to modern geometry and modern algebra, three hours; Trigonometric series, introduction to spherical harmonics, the potential function, three hours with Professor B. O. PEIRCE.—By Professor B. O. PEIRCE: Hydromechanics, two hours, first half-year.—By Professor W. F. OSGOOD: Advanced calculus, three hours; Theory of functions, three hours.—By Professor M. Bôcher: Vector analysis, three hours, first half-year; Finite differences and difference equations, three hours, second half-year.—By Professor C. L. Bouton: The elementary theory of differential equations, three hours, first half-year; Differential equations and Lie's theory, three hours.—By Professor J. L. Coolidge: Probability, three hours; Line geometry, three hours.—By Dr. D. Jackson: Infinite series and products, three hours, first half-year; Advanced algebra, three hours, second half-year; The theory of numbers including the theory of ideals, three hours.

The following advanced courses in mathematics are offered at the Italian universities during the academic year 1911–1912. Courses in algebra, analytic geometry, projective and descriptive geometry, and elementary courses in the calculus, mechanics, astronomy, and geodesy are not included:

University of Bologna: By Professor C. Arzelà: Higher mathematics, three hours.—By Professor P. Burgatti: Dynamics of rigid bodies with application to planetary motion; equilibrium of a rotating fluid mass, three hours.—By Professor L. Donati: Account of the different electromagnetic theories and of the principle of relativity, three hours.—By Professor S. Pincherle: Theory of analytic functions, linear differential equations, three hours.—By Professor U. Scarpis: Operation groups and their application to the theory of numbers, three hours.

University of Catania.—By Professor M. De Franchis: Geometry on algebraic curves and surfaces, hyperelliptic surfaces, four hours.—By Professor G. Pennacchietti: Dynamics of rigid bodies, mechanics of deformable media, four hours.—By Professor C. Severini: Integral equations and their applications to analysis, four hours.—By ———: Mathematical physics, four hours.

University of Genoa.—By Professor E. E. Levi: Elementary theory of functions of one and more complex variables, problem of uniformization of polydromic functions, four hours.—By Professor G. Loria: Algebraic and transcendental curves and surfaces, three hours.—By Professor O. Tedone: Integration methods of Riemann-Volterra, application to boundary problems, three hours.

UNIVERSITY OF NAPLES.—By Professor F. AMODEO: History of mathematics during the middle ages (XIII to XVI century), three hours.—By Professor A. DEL RE: Grassmann's analysis in n-dimensions with application to geometry and mechanics in spaces of constant curvature, four and one-half hours.—By Professor R. MARCOLONGO: Application of vector-homographics to hydrodynamics, three hours.—By Professor D. MONTESANO: General theory of algebraic surfaces, surfaces of the third and fourth order, four and one-half hours.—By Professor E. PASCAL: Selected chapters of advanced analysis, three hours.—By Professor E. PINTO: Electrostatics, four and one-half hours.—By Professor G. TORELLI: Analytic theory of numbers (advanced part), three hours.

UNIVERSITY OF PADUA.—By Professor F. D'ARCAIS: General theory of functions, elliptic functions, four hours.—By Professor U. CISOTTI: Mathematical theory of elasticity with technical applications, three hours.—By Professor P. GAZZANIGA: Theory of numbers, three hours.—By Professor T. LEVI-CIVITA: Waves in their different meanings, four and one-half hours.—By Professor G. RICCI: Absolute differential calculus with applications, four hours.—By Professor F. SEVERI: Theory of algebraic functions of two variables and of their integrals (advanced part), four hours.—By Professor G. VERONESE: Foundations of geometry (advanced part), four hours.

UNIVERSITY OF PALERMO.—By Professor G. BAGNERA: General theory of analytic functions, algebraic functions of one variable, three hours.—By Professor M. GEBBIA: Elasticity, wave theory of light, four and one-half hours.—By Professor G. B. GUCCIA: General theory of algebraic curves and surfaces, four and one-half hours.—By Professor A. VENTURI: Figure of planets and especially of the Earth, Pratt's, Stoke's, Helmert's theories, shell-tides, gravity, three hours.

UNIVERSITY OF PAVIA.—By Professor L. BERZOLARI: Birational transformations in the plane and in space, three hours.—By Professor F. GERBALDI: Functions of a complex variable, elliptic functions, three hours.—By Professor G. VIVANTI: Calculus of variations, integral equations, three hours.—By ———: Mathematical physics, three hours.

UNIVERSITY OF PISA.—By Professor E. BERTINI: Geometry on an algebraic surface (advanced part), three hours.—By Professor L. Bianchi: Arithmetic theory of quadratic (binary and ternary) forms, principles of analytic arithmetic, arithmetic of algebraic fields, four and one-half hours.—By Professor U. Dini: Spherical and Bessel's functions, four and one-half hours.—By Professor G. A. Maggi: Equilibrium and motion of elastic bodies, application to optics, four and one-half hours.—By Professor P. Pizzetti: Spherical astronomy, determination of planetary orbits, precession, nutation, theory of the tides, four and one-half hours.

University of Rome.—By Professor G. Bisconcini: Elementary differential properties of curves and surfaces, three hours.—By Professor G. Castelnuovo: Differential geometry, three hours.—By Professor G. Lauricella: Boundary problems, three hours.—By Professor L. Orlando: Mathematical and physical grounds of aerial navigation, three hours.—By Professor L. Silberstein: Principles of thermodynamics, electromagnetics and optics, mechanics according to the principle of relativity, three hours.—By Professor V. Volterra: Optics, three hours; application of mechanics to geophysical questions, three hours.

University of Turin.—By Professor T. Boggio: Equilibrium of a rotating fluid mass, three hours.—By Professor G. Fubini: Theory of partial differential equations in both the real and the complex field, Cauchy's and boundary problems, three hours.—By Professor G. Sannia: Geometrical applications of the theory of algebraic forms, one hour.—By Professor C. Segre: Continuous groups of transformations, three hours.—By Professor C. Somigliana: Propagation of heat, thermodynamics, three hours.

Professor L. Fejér, of the University of Klausenburg, has accepted a professorship of mathematics at the University of Budapest.

Dr. M. Hennequin has been appointed director of mathematical conferences of the University of Caen.

Dr. M. Plancherel, of the University of Geneva, has been appointed associate professor of mathematics at the University of Freiburg, Switzerland.

DR. W. LÁSKA, of the technical school at Lemberg, has been appointed professor of mathematics at the Bohemian University of Prague.

PROFESSOR E. SCHMIDT, of the University of Erlangen, has accepted a professorship of mathematics at the University of Breslau.

PROFESSOR G. CANTOR, of the University of Halle, has been elected a corresponding member of the royal institute of Venice.

DR. E. ZORETTI has been appointed professor of rational mechanics at the University of Caen.

PROFESSOR L. ISELY, of the Swiss University at Neuchâtel, has retired from active teaching.

PROFESSOR F. D'ARCAIS, of the University of Padua, has been elected to membership in the royal institute of Venice.

THE royal academy of sciences of Naples has recently elected Professor D. MONTESANO resident member, Professor U. DINI, of the University of Pisa, non-resident member, and Professor G. GUCCIA, of the University of Palermo, corresponding member.

PROFESSOR R. DEDEKIND, of the technical school at Brunswick, and Professor Sir J. LARMOR, of the University of Cambridge, have been elected foreign members of the royal academy dei Lincei, of Rome; Professor O. TEDONE, of the University of Genoa, has been elected to corresponding membership.

PROFESSOR T. LEVI-CIVITA, of the University of Padua, has been elected corresponding member of the academy of sciences of Paris.

THE honorary degree of doctor of science has been conferred by Colgate University upon Professor H. E. SLAUGHT, of the University of Chicago.

PROFESSOR L. G. WELD, of the University of Iowa, has resigned. Professor A. G. SMITH succeeds Professor Weld as head of the department of mathematics.

AT Dartmouth College, Professor A. D. PITCHER, of the University of Kansas has been appointed assistant professor of mathematics; Mr. MEYER GABA, of the Rolla School of

Mines, and Mr. C. R. DINES, of Harvard University, have been appointed instructors in mathematics.

AT the University of Kansas, Professor J. N. VAN DER VRIES has been elected chairman of the department of mathematics; Dr. H. N. JORDAN has been appointed assistant professor of mathematics; Miss HAZEL McGREGOR, Mr. J. J. WHEELER, and Mr. J. O. HASSLER have been appointed instructors in mathematics.

DR. H. E. BUCHANAN, of the University of Wisconsin, has accepted the professorship of mathematics at Carleton College, Northfield, Minnesota.

MR. W. E. SMITH has been appointed assistant professor of mathematics at the Pennsylvania State College.

DR. D. BUCHANAN has been appointed assistant professor of mathematics at Queen's College, Kingston, Ontario.

REV. A. S. HAWKESWORTH has resigned his lectureship in higher mathematics and semitic languages in the University of Pittsburgh.

DR. H. W. MOTT has been appointed assistant professor of mathematics at the University of Wisconsin.

DR. R. L. MOORE, of Northwestern University, has been appointed instructor in mathematics at the University of Pennsylvania.

MR. E. P. R. DUVAL has been appointed instructor in mathematics at Princeton University.

AT Middlebury College Mr. G. H. CRESSE has been appointed head of the department of mathematics with the rank of assistant professor.

MR. W. C. KRATHWOHL has been appointed instructor in mathematics in Washington University, St. Louis.

DR. J. E. ROWE will have charge of mathematical work at Haverford College during the sabbatical absence of Professor L. W. Reid.

DR. G. D. GABLE, professor of mathematics at Wooster University, died August 23 at the age of forty-eight. Professor Gable had been a member of the American Mathematical Society since 1892.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Arnoux (G.). Essai de géométrie analytique modulaire a deux dimensions. Paris, Gauthier-Villars, 1911. 8vo. 11+160 pp. Fr. 6.00.

Becker (G. F.) and Van Orstrand (C. E.). Hyperbolic functions. New edition. (Smithsonian mathematical tables.) Washington, D. C., Smithsonian Institution, 1911. 8vo. 51+321 pp. $4.00

Beltrami (E.). Opere matematiche, pubblicate per cura della facoltà di scienze della r. università di Roma. Tomo III. Milano, Hoepli, 1911. 4to. 488 pp. L. 25.00

Birckenstaedt (M.). Zwei neue allgemeine Differentiationsgesetze. (Progr.) Altona, 1911. 4to. 11 pp.

Blum (F.). Die infinitesimale Biegung von Flächen bei vollstandiger Starrheit eines Kurvensystems. (Diss.) Tubingen, 1909. 8vo. 41 pp.

Braun (W.). Bestimmung der Korperdiscriminante in einem kubischen Zahlkorper. (Diss.) Strassburg, 1909. 8vo. 54 pp.

Burgwedel (R.). Ueber die Euler'schen und Gauss'schen Methoden der Primzahlbestimmung. (Diss.) Strassburg, 1910. 8vo. 140 pp.

Burnside (W.). Theory of groups of finite order. 2nd edition. London, Cambridge University Press, 1911. 8vo. 536 pp. 15 s.

Clairin (J.). Cours de mathématiques générales. Tome 2: Calcul intégral et applications géométriques. Lille, Janny, 1911. 4to. 156 pp. Fr. 10.00

Denton (W. W.). See Young (J. W.).

Dienger (K.). Beitrag zur Lehre von den arithmetischen und geometrischen Reihen hoherer Ordnung. (Progr.) Rastatt, 1910. 4to. 16 pp.

Diesing (M.). Einführung in die Differentialrechnung und Anwendung derselben auf Maxima, Minima, unendliche Reihen und Quotienten in der unbestimmten Form $\frac{0}{0}$. (Progr.) Halle, 1911. 4to. 35 pp.

Eibenbaum (C.). Die Gleichung des Pythagoras mit der Einschränkung des Fermat. Berlin, Schultz, 1911. 4 pp. M. 3.00

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III. APPLIED MATHEMATICS.

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THE EIGHTEENTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

The eighteenth summer meeting of the Society was held at Vassar College on Tuesday and Wednesday, September 12–13, 1911, extending through two sessions on Tuesday and a morning session on Wednesday. The attendance included the following thirty-two members:

Dr. R. C. Archibald, Dr. F. W. Beal, Dr. Elizabeth R. Bennett, Professor G. D. Birkhoff, Professor Joseph Bowden, Professor W. G. Bullard, Dr. H. T. Burgess, Professor F. N. Cole, Professor L. L. Conant, Dr. Elizabeth B. Cowley, Mr. E. P. R. Duval, Professor L. P. Eisenhart, Professor T. S. Fiske, Mr. Meyer Gaba, Miss Ida Griffiths, Dr. W. A. Hurwitz, Professor C. J. Keyser, Professor W. R. Longley, Professor J. H. Maclagan-Wedderburn, Professor G. A. Miller, Dr. H. H. Mitchell, Professor G. D. Olds, Professor Arthur Ranum, Dr. J. E. Rowe, Professor C. H. Sisam, Professor A. W. Smith, Professor Virgil Snyder, Professor E. B. Van Vleck, Professor Oswald Veblen, Professor A. G. Webster, Professor H. S. White, Professor J. W. Young.

Ex-Presidents T. S. Fiske and H. S. White occupied the chair at the several sessions. The Council announced the election of the following persons to membership in the Society: Professor Frederick Anderegg, Oberlin College; Dr. C. E. Brooks, Northwestern University; Mr. G. G. Brower, Cascadilla School; Mr. W. C. Graustein, Harvard University; Dr. Dunham Jackson, Harvard University; Mr. W. V. Lovitt, University of Washington; Mr. J. C. Rayworth, Washington University; Mr. L. L. Smail, University of Washington; Dr. E. B. Stouffer, University of Illinois; Dr. S. E. Urner, University of Wisconsin; Professor J. N. Van der Vries, University of Kansas; Mr. C. W. Wester, University of Washington. Twelve applications for membership in the Society were received.

On both days of the meeting luncheon was served in the College buildings. On Tuesday evening twenty-nine of the members gathered at the customary informal dinner, at the close of which Professor Birkhoff spoke briefly on Moore's general analysis and Professor Webster on wider views in mathematics and physics. Wednesday afternoon was devoted to an excursion to Lake Mohonk. At the close of the meeting the hospitality of Vassar College was gratefully acknowledged by a vote of thanks.

During the summer the following committees have been appointed: on nomination of officers and members of the Council, Professors Bôcher, Dickson, and Morley; on arrangements for the summer meeting and colloquium at Madison in 1913, Professors Moore, Osgood, Van Vleck, and the Secretary.

The following papers were read at the summer meeting:

(1) Professor EDMUND LANDAU: "Ueber eine idealtheoretische Funktion."

(2) Dr. W. A. HURWITZ: "On the pseudo-resolvent to the kernel of an integral equation."

(3) Dr. W. A. HURWITZ: "On mixed linear integral equations."

(4) Dr. ELIZABETH R. BENNETT: "Simply transitive primitive groups whose maximal subgroup contains a transitive constituent of order p^2 or pq, or a transitive constituent of degree 5."

(5) Professor FLORIAN CAJORI: "On a rare book of Michel Rolle and the history of 'Rolle's theorem.'"

(6) Professor L. C. KARPINSKI: "The algebra of Abū Kāmil Shojā ben Aslam."

(7) Dr. F. W. BEAL: "Normal congruences determined by centers of geodesic curvature."

(8) Professor ARNOLD EMCH: "On the congruence of rays realizing circular transformations between two planes."

(9) Professor JOSEPH BOWDEN: "The two fundamental relations of the calculus" (preliminary communication).

(10) Dr. J. E. ROWE: "Covariant curves of the R^4 and R^5."

(11) Professor G. A. MILLER: "A third generalization of the groups of the regular polyhedra."

(12) Professor G. A. MILLER: "Some properties of the group of isomorphisms."

(13) Professor L. P. EISENHART: "Minimal surfaces in plane four-space."

(14) Professor ARTHUR RANUM: "On the projective differential geometry of spreads generated by ∞^1 flats."

(15) Mr. E. W. CASTLE: "A graduation of the combined experience table of mortality to Makeham's formula by the method of moments."

(16) Dr. S. LEFSCHETZ: "On the existence of loci with given singularities."

(17) Dr. S. Lefschetz: "On some topological properties of plane curves."

(18) Professor Virgil Snyder: "Periodic quadratic transformations in a ternary field."

(19) Professor A. G. Webster: "On a new mixed boundary problem in connection with the telegrapher's equation."

(20) Professor A. G. Webster: "On the wave potential of a circular ring of sources."

(21) Professor A. G. Webster: "Solid viscosity versus elastic hysteresis in the transverse vibration of an elastic bar."

(22) Professor G. D. Birkhoff: "New proof of a theorem concerning matrices of analytic functions."

(23) Professor G. D. Birkhoff: "On the simplest type of irregular singular point."

(24) Professor G. A. Bliss: "A generalization of the preparation theorem for a power series in several variables."

(25) Professor Oswald Veblen: "On the definition of multiplication of irrational numbers."

(26) Dr. H. T. Burgess: "One-parameter groups of contact transformations defined on a fixed quadric by a bilinear form."

(27) Professor Joseph Bowden: "Making a recitation schedule."

Mr. Castle's paper was communicated to the Society by Professor Glover. Dr. Lefschetz was introduced by Professor Webster. The papers of Professors Landau, Cajori, Karpinski, Emch, Miller (second paper), Mr. Castle, and Professor Bliss were read by title. Professor Landau's paper will appear in the October number of the *Transactions*. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

2. The method developed by Fredholm for the solution of the linear integral equation is comparatively simple in case the homogeneous equation admits no solutions not identically zero; a resolvent function to the kernel of the equation can be expressed in terms of the determinant and the first minor. If the homogeneous equation admits non-zero solutions, the development of necessary and sufficient conditions for the existence of solutions of the non-homogeneous equation by Fredholm's method, while similar in character to the simpler case, involves the introduction of minors of higher order, thereby necessitating somewhat tedious and complicated algebraic manipulation.

For the sake of simplicity of presentation, it seems valuable to secure these results more simply. Dr. Hurwitz's first paper, by the artifice of transforming a given integral equation into a new one for which the corresponding homogeneous equation has no solutions other than zero, proves, without the introduction of minors of higher order, the following theorems, furnishing the theory desired:

1) If the kernel $K(x, y)$ is continuous, the equations

$$\varphi(x) = \int_a^b K(x, s)\varphi(s)ds, \tag{1}$$

$$\psi(x) = \int_a^b \psi(s)K(s, x)ds \tag{1'}$$

have each a finite number of linearly independent continuous solutions, and this number is the same for the two equations.

2) If the complete sets of normal orthogonal continuous solutions of (1), (1′) are respectively $\varphi_1(x), \varphi_2(x), \ldots, \varphi_m(x)$; $\psi_1(x), \psi_2(x), \ldots, \psi_m(x)$, there exists a continuous function $k(x, y)$, the pseudo-resolvent to the given kernel $K(x, y)$, such that

$$k(x, y) = K(x, y) + \int_a^b K(x, s)k(s, y)ds - \sum_{i=1}^{m} \psi_i(x)\psi_i(y),$$

$$k(x, y) = K(x, y) + \int_a^b k(x, s)K(s, y)ds - \sum_{i=1}^{m} \varphi_i(x)\varphi_i(y).$$

3) A necessary and sufficient condition that the equation

$$u(x) = f(x) + \int_a^b K(x, s)u(s)ds \tag{2}$$

(where $f(x)$ is continuous) have a continuous solution, is that

$$\int_a^b \psi_i(x)\, f(x)\, dx = 0 \quad (i = 1, 2, \ldots, m).$$

If this condition is fulfilled, the most general solution of (2) is

$$u(x) = f(x) + \int_a^b k(x, s)f(s)ds + \sum_{i=1}^{m} c_i\varphi_i(x),$$

where $c_1, c_2, \ldots, c_m$ are constants.

3. By a mixed integral equation is understood one which involves the unknown function under several integral signs, operating over regions of different numbers of dimensions, and which also involves the value of the unknown function at isolated points. Dr. Hurwitz's second paper seeks to indicate how the theory of mixed linear integral equations follows the laws governing linear equations in general, by the establishment of the following theorem, bearing on an especially simple case:

Let $f(x)$, $k_1(x)$, $k_2(x)$, $\cdots$, $k_m(x)$ be continuous functions of x, $a \leqq x \leqq b$, and $K(x, y)$ a continuous function of x and y, $a \leqq x \leqq b$, $a \leqq y \leqq b$; let x_1, x_2, $\cdots$, x_m lie in the interval $a \leqq x \leqq b$. Then if the homogeneous mixed linear integral equation

$$u(x) = \sum_{i=1}^{m} k_i(x)u(x_i) + \int_a^b K(x, s)u(s)ds$$

has no continuous solution, $a \leqq x \leqq b$, other than $u(x) \equiv 0$, the equation

$$u(x) = f(x) + \sum_{i=1}^{m} k_i(x)u(x_i) + \int_a^b K(x, s)u(s)ds$$

has one and only one continuous solution.

4. Certain restrictions on the transitive constituents occurring in the maximal subgroup of a simply transitive primitive group when the maximal subgroup contains a transitive constituent of degree 3 or 4 or a transitive constituent of order p, have been determined. Dr. Bennett considers how the degree of a simply transitive primitive group and the transitive constituents in its maximal subgroup are limited when the maximal subgroup contains a transitive constituent of order p^2 or pq or a transitive constituent of degree 5.

5. Professor Cajori's paper describes Rolle's Démonstration d'une méthode pour résoudre les égalitez de tous les degrez, etc., a book which has not been previously described by historians. Recent doubts as to whether "Rolle's theorem" is due to Rolle are removed by this book, which gives the theorem in the course of the demonstration of Rolle's "method of cascades." The book gives also the solution of equations by the intersections of circles and parabolas, and a discussion of quadratic residues containing proofs of the theorem of Diophantus

that no integer of the form $4n + 3$ can be the sum of two squares and of the theorem that no integer $8n + 7$ can be the sum of three squares. "Rolle's theorem" apparently was not named after Rolle until after the middle of the nineteenth century. It rose to prominence after O. Bonnet based upon it a proof of the theorem of mean value (Serret, Calcul différentiel et intégral, 1868, page 17) and Taylor's theorem came to be proved by the mean value theorem (Homersham-Cox and Rouché).

6. In 1841 Chasles called attention to the Latin translation of an Arabic Algebra which is found in the Paris manuscript 7377A. This Algebra is now known to be the work of Abū Kāmil Shojā ben Aslam, who flourished during the early tenth century. Two commentaries on this Algebra were written within the same century and an Arabic historian, Ibn Khaldūn (1332–1406) states that a Spanish commentary was written by one Al-Korekhi, otherwise unknown. Mordechai Finzi (c. 1460) of Mantua made a Hebrew translation which bears internal evidence that it was based upon a Spanish version. Up to the present time copies have been found of the Latin and Hebrew translations only.

Professor Karpinski presents the first study which has been made of this Algebra since Chasles called attention to it. This shows that Abū Kāmil devised several new geometrical solutions of the three types of quadratic equations ($x^2 + bx = c$, $x^2 + c = bx$, $x^2 = bx + c$) but that otherwise he followed closely the lines laid down by Al-Khowarizmi and that Leonard of Pisa drew many of his algebra problems, complete with their solutions, from Abū Kāmil.

7. In 1892 Caronnet announced the following theorem: The necessary and sufficient condition that the lines which join the centers of geodesic curvature of the curves of an orthogonal system on a surface shall form a normal congruence is that the corresponding radii of geodesic curvature be functions of one another or that the curves in one family or both have constant geodesic curvature. Dr. Beal's paper is concerned with the determination of normal congruences of this kind and with an investigation of their properties. For the sake of brevity he has referred to them as associated normal congruences.

When both radii of geodesic curvature are constant the associated normal congruence is the congruence of normals of a Bianchi transform of the given surface. In this case the given surface must be pseudospherical and of total curvature equal to the negative of the sum of the squares of the constant geodesic curvatures. Associated normal congruences exist for any surface for any given functional relation between the radii of geodesic curvature and are determined by a partial differential equation of the first order, which is also of the first degree when and only when one radius is equal to a constant or to a constant times the other. The condition in the theorem due to Caronnet that the system of curves be orthogonal may be replaced by the condition that the curves of the two families intersect under a constant angle. The congruences determined by these systems will be associated normal congruences.

8. As is well known, the Steinerian transformation realizes in a very simple geometric manner a general uniform quadratic transformation between two planes. This correspondence is obtained by the intersection of the congruence of rays through any two skew lines with those planes. The special case of circular transformations is obtained by assuming in a definite manner two of the fundamental points in each plane as the circular points at infinity. In this case the construction in the Steinerian sense becomes illusory. It is however possible to establish such a congruence by means of two inversions in space and it is the purpose of Professor Emch's paper to investigate the properties of this congruence and its limiting surface, which is of the fourth order.

9. In order to give a clear understanding of the fundamental theorems of the integral calculus Professor Bowden finds it useful to employ the symbols D and I respectively for the relations "is the derivative of" and "is an integral of." Thus $2x\,D\,x^2$, $x^2+4\,I\,2x$, $(uIv)=(vDu)$. Each of these relations is the converse of the other. Each is non-reflexive, non-symmetric, non-transitive. Among the fundamental laws are: $(uDv)\supset(-uD-v)$, $(u\,Dv)\frown(wDs)\supset(u+w)D(v+s)$, $(uDv)\frown(wDs)\supset(us+vw)D(vs)$, $(uIv)\supset(-u\,I-v)$, $(uIv)\frown(wIs)\supset(u+w)I(v+s)$, $(uIv)\frown(wIs)\supset uwI(vw+us)$.

10. In his dissertation, which appeared in the July *Transactions*, Dr. Rowe has considered certain covariant curves of the

R^4 which are combinants of two binary quartics cut out by two secant lines of the R^4; these curves are expressible in terms of three-rowed determinants of the type $|abx|$ and are called combinant curves of the R^4. In Section I of the present paper certain covariant loci of the R^4 are obtained which are expressible in terms of three-rowed determinants of the types $|abc|$ and $|abx|$; these are not combinant curves as defined above and for convenience are called bicombinant curves of the R^4.

The conic $A - 16B = 0$ (conic on flexes of R^4) cuts out two other points besides the flexes which we call q_i; these two points determine a line L_4 which determines two other points r_i. L_4 is defined as a member of a pencil of bicombinant lines as follows:

$$(1) \qquad 5L_4 = L_3 + 4(Q_1^2 - 4Q_0Q_2)L_1,$$

where L_1 cuts out the two q_i on $A - 12B = 0$ (envelope of flex lines of cubic osculants of the R^4) and is the locus such that tangents to the R^4 from it have their parameters apolar to the flexes; the expression in the parenthesis of (1) is the condition for the q_i to unite; and L_3 is a locus such that tangents from it to R^4 have their parameters apolar to the q_i each taken three times. The equation of the line L_4 and the quadratic giving the r_i are derived in this way. The condition that the q_i and r_i be apolar is the discriminant of the conic $A - 16B = 0$.

Two bicombinant conics are given: the conic on the nodes and q_i, and the conic on the nodes and r_i.

The bicombinant cubic on the flexes and nodes of the R^4 is written out in full. By mere substitutions in this equation are obtained the following bicombinant cubics: on nodes and q_i each taken three times; on nodes, q_i each taken twice, and on r_i; on nodes, q_i, and on r_i each taken twice; on nodes, and r_i each taken three times.

In Section II the combinant curves of the R^5 of orders two and three are considered. The R^5 has a pencil of combinant conics having double contact on the covariant line. The combinant cubics of the R^5 are expressed in terms of four among which no linear relation exists in general. There is a unique combinant cubic expressed in terms of the above four which cuts out the flexes of the R^5 and six other points.

Also, a method is given for finding a certain set of invariants for any rational curve of odd order by reason of the R^{2n+1} having a covariant rational line quartic whose invariants are known. These invariants are actually found for the R^5.

11. The abstract properties of the regular polyhedron groups were first developed by Sir W. R. Hamilton in a paper presented to the Royal Irish Academy in 1856. These groups are defined by three equations between the two generators s_1, s_2. The equations may be written in the form

$$s_1^\alpha = s_2^\beta = (s_1s_2)^\gamma = 1,$$

where two of the three positive integers α, β, γ are equal to 2 and 3, while the third has the value 3, 4, or 5, according as the group is tetrahedral, octahedral, or icosahedral, respectively. In the present paper Professor Miller considers the possible groups when the last condition is omitted. That is, he considers the groups generated by s_1, s_2 when these operators satisfy two conditions of the form

$$s_1^\alpha = s_2^\beta = (s_1s_2)^\gamma$$

with the same restrictions on α, β, and γ as before.

It is proved that no more than four non-abelian groups can satisfy such a pair of conditions, and that the present generalizations lead to simpler systems of groups than the two earlier generalizations considered by the same author. The following theorems were proved: If two non-commutative operators satisfy the conditions $s_1^3 = s_2^3 = (s_1s_2)^2$, they generate either the tetrahedral group or the non-twelve group of order 24; if they satisfy the conditions $s_1^3 = s_2^4 = (s_1s_2)^2$, they generate either the octahedral group or a group of order 48 known as G_{52}; if they satisfy the two conditions $s_1^3 = s_2^5 = (s_1s_2)^2$, they generate either the icosahedral group or a group of order 120 known as G_{120}. In all the other possible cases the non-abelian groups are either the direct products of one of the given groups and a cyclic group or an isomorphism between such groups.

12. The first part of Professor Miller's second paper is devoted to a study of the group of isomorphisms I of an abelian group G. It is pointed out that I can always be represented as a transitive substitution group on letters corresponding to a set of operators of G and that this can be done in only one way. When I is represented as such a substitution group it is very easy to prove that unless G is of order 2^m and of type (1, 1, 1, . . .) its group of isomorphisms cannot be represented as a substitution group on letters corresponding to a set of subgroups of G.

The principal theorems proved in the second part of this paper may be stated as follows: The group of isomorphisms of a non-abelian group cannot be represented transitively on letters corresponding to a set of commutative operators of this non-abelian group. If the group of isomorphisms I of a group G is represented on letters corresponding to a set of operators of G, then I cannot be more than doubly transitive unless it is the symmetric group of order 6. If I is doubly transitive, and is not this symmetric group, then the subgroup composed of all the substitutions of I which omit one letter is always imprimitive.

13. At the April meeting of the Society Professor Eisenhart established a method of giving to the parametric equations of any minimal curve in plane four-space a form analogous to the Weierstrass equations of minimal curves in ordinary space. In the present paper he makes use of these forms in order to define and discuss minimal surfaces (dimensional spreads) in plane four-space. This form is of particular value in the study of algebraic surfaces. The geometry of a general minimal surface in the neighborhood of a point is studied to advantage. With any such minimal surface it is possible to associate a family of applicable surfaces analogous to the group of associate minimal surfaces in ordinary space and therefrom follow formulas analogous to the formulas of Schwarz, with the consequent theory concerning the determination of a minimal surface by a strip. Reference is made to the minimal surfaces in four-space, defined by an equation of the form $z + it = f(x + iy)$, where f is an analytic function; these surfaces have been considered from another point of view by Karl Kommerell under the title Riemann Surfaces. This paper will be published in the *American Journal of Mathematics.*

14. In this paper Professor Ranum generalizes to n-dimensional space the results of his paper on "Ruled surfaces and planed hypersurfaces in four-dimensional space," of which an abstract is given in the BULLETIN, volume 17, page 302. An m-dimensional spread S_m generated by ∞^1 flats (linear spaces) is given by a set of homogeneous parametric equations. Its tangent spread S_{m+r}, is defined as the locus of the connecting flats of its consecutive generators. The focal spread of S_m, namely the locus of the flats of intersection of its consecutive

generators, is shown to be an S_{m-r}. The number r is called the rank of S_m. S_{m+r} and S_{m-r} are in general of lower rank than S_m. The focal spread of the tangent spread of S_m is in general a larger spread enclosing S_m. Calling the focal spread of the focal spread the 2nd focal spread, etc., the author shows that the 1st, 2nd, . . . , ith focal spreads of the ith tangent spread of S_m are all of the same rank and all enclose S_m. Various other projective differential properties of these spreads are derived, the principle of duality is employed, and a general scheme is obtained for classifying the spreads.

15. The chief purpose of Mr. Castle's paper is to present a graduation of the combined experience of actuaries table of mortality to Makeham's formula by the method of moments. The different methods of graduating a mortality table may be roughly classified under two heads: those in which some formula is assumed to express the functional relation between the number living and the age x, the arbitrary constants involved being determined from the original data; and, second, those in which the number living at any age x is derived from the original data in such a way as to produce a smooth curve without satisfying any functional relation.

In the graduation here presented Makeham's formula

$$y = ks^x g^{c^x}$$

was assumed to express the functional relation between the number living y and the age x, the arbitrary constants k, s, g, and c being determined by the method of moments. Briefly, the theory involved in graduation by the method of moments is that in fitting an assumed formula to statistical data the values of the constants which depend upon all the given data are obtained by finding the area and moments of the theoretical curve, and equating them to the corresponding area and moments as obtained from the statistical data.

The work has been carried out in considerable detail with the secondary object of being of use to those who desire to employ the method of moments in graduating similar statistical data. The paper will be published in the *Annals of Mathematics.*

16. The main point of Dr. Lefschetz's first paper in the introduction of a new inequality which the Plückerian numbers of any

algebraic multiplicity have to satisfy. Plane curves are chiefly treated. Adopting Salmon's notations, since a double point requires one condition, a cusp two conditions to be satisfied by the coefficients of the general curve of a given order m, we must have

$$\delta + 2k \leqq \frac{m(m+3)}{2} - 8$$

for such conditions are expressed evidently by the vanishing of as many invariants. A discussion follows, and maxima are given for the number of cusps alone. The question of the maximum of cusps for a given deficiency is also treated. Finally the same method is applied to surfaces. The cases of surfaces of the third, fourth, and fifth orders are treated at some length, and for them maxima for the number of binodes found. To mention one interesting result, it is found that a surface S_4 can have no binodes if it has more than 14 nodes, and never more than 8 binodes.

17. The chief feature of Dr. Lefschetz's second paper is an application of inversion to a descriptive study of curves. First, some well known properties of open or closed branches receive new proofs. Next a study of curvature follows with an intended application to inflexions of certain branches by means of an inversion. Then a detailed study is made of closed branches having one single multiple point of multiplicity k. A representation of such a system is given by means of a sketch, the chief feature of which is the formation of certain polygons having for vertices those of a regular polygon of k sides. By means of some simple properties of such polygons, a theorem is proved for the case of $k = 2n + 1$, which is then shown to be equivalent to the theorem of Möbius giving 3 as the minimum of inflexions of an odd branch without any multiple points. Really the idea is to substitute for a curve having infinite branches one lying at finite distance. Möbius did this by projecting plane curves on a sphere; the author obtains the same results by inverting the curve with respect to a pole through which it does not pass.

18. In the study of finite groups of Cremona transformations extensive use is made of the composition of each generator in terms of quadratic transformations. If two fundamental points are identical with two fundamental points of the inverse

system the iteration of the quadratic transformation is also quadratic (or the identity). Starting from the theorem that in a cyclic collineation a point and its successive images lie on a conic, Kantor drew similar conclusions concerning quadratic transformations. Professor Snyder showed that neither one of these theorems is true unless restrictions are imposed on the coefficients, and that in consequence extensive categories of real transformations are omitted in Kantor's classification.*

19. If u and v stand for the current and potential respectively multiplied by an exponential factor, each satisfies the equation

$$\frac{\partial^2 u}{\partial^2 x} - \frac{\partial^2 u}{\partial y^2} + u = 0$$

and the problem treated in Professor Webster's first paper is the satisfaction of Cauchy's conditions, function and conormal derivative being given for $y = 0$, $x \geqq 0$, while for $x = 0$, $y \geqq 0$ the condition is

$$a_0 u + a_1 \frac{\partial u}{\partial y} + a_2 \frac{\partial^2 u}{\partial y^2} + \cdots + a_n \frac{\partial^n u}{\partial y^n}$$

$$+ b_0 v + b_1 \frac{\partial v}{\partial y} + b_2 \frac{\partial^2 v}{\partial y^2} + \cdots + b_m \frac{\partial^m v}{\partial y^m} = \psi(y).$$

By means of Riemann's method, integration around the proper contour leads to a Volterra integro-differential equation for the conormal derivative, which gives the solution by the usual formula.

20. In Professor Webster's second paper the solution of the wave equation is found in the form of a definite integral, which is developed into two series for practical computation.

21. In Professor Webster's third paper the theory of the normal vibrations of a bar vibrating transversely is worked out according to Voigt's theory of viscosity, and also according to the theory of heredity as developed by Boltzmann, Wiechert, and especially Volterra, and the results contrasted. Experiments favor the heredity theory.

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*S. Kantor, "Premiers fondements pour une théorie des transformations périodiques univoques," *Atti della R. Accademia . . . di Napoli*, ser. 2, vols. 3 and 4 (1891), pp. 1–356.

22. In this note Professor Birkhoff employs the theory of integral equations to derive another proof of a theorem which was stated by him in an earlier paper on singular points of ordinary linear differential equations (*Transactions* of the Society, volume 4 (1908), pages 436–470).

23. In a second note Professor Birkhoff makes a detailed study of the irregular singular point of rank zero for a linear differential equation of the second order. The method depends on the theorem of the previous note.

24. The paper of Professor Bliss contains a generalization of a well-known theorem concerning a power series in several variables $F(x_1, x_2, \cdots, x_m; y)$. This theorem states that if $F(0, 0, \cdots, 0; y)$ begins with a term of degree v in y, then there exists a unique series $M(x_1, x_2, \cdots, x_m; y)$ with constant term different from zero, such that the product MF is a polynomial in y of the form

$$MF = y^v + a_1 y^{v-1} + a_2 y^{v-2} + \cdots + a_{v-1} y + a_v,$$

where the coefficients a_k are power series in $x_1, x_2, \cdots, x_m$ which have no constant terms. Evidently in a sufficiently small region about the origin the roots of the function F are all roots of the polynomial, and vice versa.

The generalization has to do with a system of functions $F_\alpha(x_1, x_2, \cdots, x_m; y_1, y_2, \cdots, y_n)$ $(\alpha = 1, 2, \cdots, n)$ for which the series $F_\alpha(0, 0, \cdots, 0; y_1, y_2, \cdots, y_n)$ begin with so-called "characteristic" polynomials $f_\alpha(y_1, y_2, \cdots, y_n)$, homogeneous and of degrees v_α, respectively. Under certain restrictions on the coefficients of the functions f_α, there will then always exist a polynomial

$$P(x_1, x_2, \cdots, x_m; y_n) = y_n^N + a_1 y_n^{N-1} + \cdots + a_{N-1} y + a_N$$

of degree $N = v_1 v_2 \cdots v_n$, such that every root of the equations $F_\alpha = 0$ in a sufficiently small region about the origin makes the polynomial vanish, and conversely to every root of the polynomial there corresponds a root of the equations. The polynomial is linearly expressible in the form

$$P = M_1 F_1 + M_2 F_2 + \cdots + M_n F_n,$$

where the coefficients M_α are series in $x_1, x_2, \cdots, x_m; y_1, y_2, \cdots, y_n$.

25. In the genetic theory of the irrational numbers, each open cut in the scale of rationals determines an irrational. Arbitrary definitions are given of the operations of addition and multiplication, which are then proved to satisfy the same laws of combination (associative, distributive, commutative, inverse) as the rational numbers. Professor Veblen showed that it is possible, without altering the definition of addition, to give another definition of multiplication such that all the laws of combination hold good, such that the product of two rationals or of a rational and an irrational is the same as in the ordinary system, but such that the product of two irrationals is different from their product in the ordinary system. The new definition depends on the well-ordering of the continuum, and an infinity of similar definitions may be set up. The resulting number systems have applications in the foundations of geometry.

26. Denote the fixed quadratic surface by Q, its equation by $Q(x) = 0$, the polar form of $Q(x)$ by $Q(x, y)$, and the general bilinear form by $B(x, y)$; and write the equations

$$(1) \qquad Q(x) = 0, \qquad (2) \qquad Q(y) = 0,$$

$$(3) \qquad R(x, y, \lambda) \equiv \lambda Q(x, y) + B(x, y) = 0.$$

Since the region of definition of the variables is limited to Q by (1) and (2), the equations (1), (2), and (3) considered together set up on Q a family of point-conic correspondences which are denoted by $\Gamma(x, y, \lambda)$.

For a fixed value of λ, to any two consecutive points on Q there correspond by $\Gamma(x, y, \lambda)$ two consecutive conics on Q; the two consecutive points determine a line element on Q and the two consecutive conics determine in general two line elements on Q. As λ varies, $\Gamma(x, y, \lambda)$ establishes a system of contact transformations on Q which are denoted by $T(\lambda)$.

In this paper, Dr. Burgess proves the following: Provided $B(x, y) = 0$ is not a null system, a necessary condition that $T(\lambda)$ be a group is that the λ-matrix of $\lambda Q(x, y) + B(xy)$ possess one of the following characteristics: $[1(111)]$, $[1^\circ(111)]$, $[1(1^\circ 1^\circ 1^\circ)]$, $[(211)]$, $[(2^\circ 1^\circ 1^\circ)]$.

27. For making a recitation schedule Professor Bowden uses several relation and operation tables, of which the principal ones are (1) a sequence card combining the hours of the week

into sequences, (2) a course table showing what teachers give and what pupils wish to take the various courses, (3) a conflict table showing the common members of each pair of courses, if their number is not more than a given number, say five, and, if more, showing that fact. The first of these tables is a unary operation table, the second a dyadic relation table, the third a combination of a binary operation table and a dyadic relation table.

The course table is cut into strips showing the class lists, which are used to form the conflict table. This table is then cut into strips, one for each course, which are then easily assembled into groups, so as to have as few conflicts as possible. The courses of any one group are placed at the same sequence of hours.

The form in which the schedule first appears is itself a dyadic relation table, the left-hand column containing the department names, the top row containing the numbers of the sequences, and the intersections of rows and columns the course numbers.

F. N. COLE,
Secretary.

A NECESSARY AND SUFFICIENT CONDITION FOR THE UNIFORM CONVERGENCE OF A CERTAIN CLASS OF INFINITE SERIES.

BY DR. N. J. LENNES.

(Read before the American Mathematical Society, February 25, 1911.)

ALL numbers x such that $a \leqq x \leqq b$ constitute the interval ab. A function $f(x)$ is said to be continuous on an interval ab if (1) it is continuous in the ordinary sense for every value of x, $a < x < b$, (2) if it has right hand continuity for $x = a$ and left hand continuity for $x = b$.

THEOREM. *If on the interval ab*

$$\sum_{n=0}^{\infty} U_n(x) = f(x),$$

$U_i(x)$ $(i = 0, \cdots, \infty)$ and $f(x)$ being continuous on the interval ab, then in order that $\sum_{n=0}^{\infty} U_n(x)$ shall be uniformly convergent on

the interval ab it is necessary and sufficient that for any x_i *on ab, any arbitrary number* δ *(however small) and any arbitrary integer* N *there is an integer* $N'(i, \delta, N)$* *greater than* N*, which satisfies the following condition:*

On the interval $x_i - \delta, x_i + \delta$ *there is a value* x_i' *of* x *such that for every* $N_i'' > N'(i, \delta, N)$

$$\left| \sum_{n=0}^{N'(i,\delta,N)} U_n(x_i') - f(x_i') \right| \geqq \left| \sum_{n=0}^{N_i''} U_n(x) - f(x) \right|$$

for every value of x *on the interval* $x_i - \delta, x_i - \delta$ *which also lies on ab.*†

Proof: (1). *The condition is necessary.* (In the proof $N'(i, \delta, N)$ is written N_i'.)

By hypothesis $\sum_{n=0}^{\infty} U_n(x)$ is uniformly convergent on ab and hence on every subinterval $x_i - \delta, x_i + \delta$ which is contained in ab. Two cases are possible:

(*a*) $U_n(x) \equiv 0$ for every x on $x_i - \delta, x_i + \delta$ and for every value of n greater than some fixed number N_i. In this case the conclusion of the proposition holds for x_i' any x on $x_i - \delta, x_i + \delta$, the two differences

$$\sum_{n=0}^{N_i'} U_n(x_i') - f(x_i') \quad \text{and} \quad \sum_{n=0}^{N_i''} U_n(x) - f(x)$$

being zero.

(*b*) There is no number N_i' such that for every $n > N_i'$ $U_n(x) \equiv 0$ for x on $x_i - \delta, x_i + \delta$.

In this case for every N_i' there are values of $n, n' > N_i'$ and values $\bar{x}_i$ of x on $x_i - \delta, x_i + \delta$ such that

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* The subscripts i, δ, N indicate that $N'(i, \delta, N)$ depends on all of these.

† A special case of this lemma when

$$\sum_{n=0}^{\infty} \left| U_n(x) \right|$$

is convergent is given for the sufficient condition by Pringsheim, *Mathematische Annalen*, vol. 44 (1893), p. 82, and by Bôcher, BULLETIN, vol. 4 (1898), p. 368. The theorem of the present paper, besides stating a necessary as well as a sufficient condition, covers a larger class of functions than is covered by the theorem of Pringsheim. Thus the uniform convergence over a certain interval of an alternating series representing a continuous function follows as a special case provided that for every value x_i of x of the interval there is a neighborhood of x_i on which the numerical values of the terms decrease after a certain term.

$$(1) \qquad \left| \sum_{n=0}^{n'} U_n(\overline{x_i}) - f(\overline{x_i}) \right| > 2\epsilon,$$

where ϵ is some positive number.

The series $\sum_{n=0}^{\infty} U_n(x)$ being uniformly convergent on $x_i - \delta$, $x_i + \delta$, there is a value of n, n'' such that for all values of n, $\bar{n} > n''$,

$$(2) \qquad \left| \sum_{n=0}^{\bar{n}} U_n(x) - f(x) \right| < \epsilon.$$

Since for any value of k, $\sum_{n=0}^{k} U_n(x)$ and $f(x)$ are both continuous functions on $x_i - \delta$, $x_i + \delta$ there is a definite value of x for which

$$\left| \sum_{n=0}^{k} U_n(x) - f(x) \right|$$

reaches a maximum on the closed interval $x_i - \delta$, $x_i + \delta$. Denote this maximum by $D(k)$. Since $n'' - n'$ is a finite number, there is a value of n, $n = N_i'$, $n' \leqq N_i' \leqq n''$ for which $D(k)$ reaches the maximum for the values of k, $n' \leqq k \leqq n''$.

Let x_i' be a value of x for which

$$\left| \sum_{n=0}^{N_i'} U_n(x) - f(x) \right|$$

reaches its maximum. Then by (1) and (2)

$$\left| \sum_{n=0}^{N_i'} U_n(x_i') - f(x_i') \right| \geqq \left| \sum_{n=0}^{N_i''} U_n(x) - f(x) \right|$$

for every x on $x_i - \delta$, $x_i + \delta$ and every $N_i'' > N_i'$, which proves the condition necessary.

(2) *The condition is sufficient.* Since by hypothesis

$$\sum_{n=0}^{\infty} U_n(x) = f(x),$$

the function $f(x)$ being defined on ab, it follows that $\sum_{n=0}^{\infty} U_n(x)$ is convergent for every value of x on this interval. Hence for any preassigned positive number ϵ and any value x_i of x on ab

there is an integer N_i such that

$$\left| \sum_{n=0}^{N_i'} U_n(x_i) - f(x_i) \right| < \epsilon \tag{3}$$

for every $N_i' \geqq N_i$.

The function $\sum_{n=0}^{N_i'} (U_n(x)$ being continuous in the neighborhood of $x = x_i$ for every value of N_i', there is a positive number δ', depending on ϵ, x_i, and N_i', such that for $|x - x_i| < \delta'$

$$\left| \sum_{n=0}^{N_i'} U_n(x) - \sum_{n=0}^{N_i'} U_n(x_i) \right| < \epsilon \tag{4}$$

for any preassigned $N_i' > N_i$.

Since $f(x)$ is continuous for $x = x_i$, there is a δ'' such that for $|x - x_i| < \delta''$

$$|f(x) - f(x_i)| < \epsilon. \tag{5}$$

Let δ''' be the smaller of δ' and δ''. Then for x such that $|x - x_i| < \delta'''$ we have from (3), (4), (5)

$$\left| \sum_{n=0}^{N_i'} U_n(x) - f(x) \right| < 3\epsilon \tag{6}$$

for any arbitrarily assigned N_i'.

But by hypothesis, for every x_i and an arbitrary N_i there is an N_i' such that for any $N_i'' > N_i'$ and for x on some interval $x_i - \delta''''$, $x_i + \delta''''$ there is an x_i' such that

$$\left| \sum_{n=0}^{N_i'} U_n(x_i') - f(x_i') \right| \geqq \left| \sum_{n=0}^{N_i''} U_n(x) - f(x) \right|. \tag{7}$$

Let δ be the smaller of the two numbers δ''' and δ''''. Since (6) holds for any value of x on $x_i - \delta$, $x_i + \delta'$ it holds for $x = x_i'$. Hence if the N_i' of (7) is chosen as the N_i' of (6) it follows from (7) that for every $N_i'' > N_i'$

$$\left| \sum_{n=0}^{N_i''} U_n(x) - f(x) \right| < 3\epsilon,$$

provided $|x - x_i'| \delta$. That is $\sum_{n=0}^{\infty} U_n(x)$ is uniformly convergent on the interval $x - \delta$, $x + \delta$, or on that part of such interval which is an ab. In this manner for every point x_i on ab we obtain an interval of which x_i is an internal point (unless x_i is a or b, in which case x_i is the end point of such interval) on which $\sum_{n=0}^{\infty} U_n(x)$ is uniformly convergent.

By the Heine-Borel theorem* there is a finite subset of these intervals which completely covers the interval ab. Hence it follows by obvious considerations that $\sum_{n=0}^{\infty} U_n(x)$ is uniformly continuous on the interval ab.

COLUMBIA UNIVERSITY,
February 1, 1911.

PROJECTIVE GEOMETRY.

Projective Geometry. By OSWALD VEBLEN and JOHN WESLEY YOUNG. Volume I. Ginn and Company, Boston, 1910. x + 342 pp.

IN discussing this book, two facts must be constantly borne in mind: the authors knew exactly what they wanted to do, and they were entirely competent to do it. Here is no question of a slovenly work, compiled with no visible object, differing from an honorable line of progenitors only in the smallest details. The book is eminently one with a consistent purpose. Agree with the authors you may not; but if you do not fully enter into their point of view, you will come off second best in the argument.

The authors' main object is to build up projective geometry upon a system of independent axioms. Such a task is, certainly, not new. The names of Pieri, Vahlen, Schur, and others suggest themselves immediately when the question of projective assumptions is raised. Nor is there even any novelty in writing a students' textbook which starts from the ground. We have such a book already in the beautiful treatise of Enriques, and, in fact, we may almost say that this is the standard way to write a textbook of projective geometry today. No, the novelty consists in the set of axioms which the authors have used. These are equally serviceable for a geometry of a finite number of points, a rational geometry, a three-dimensional continuum, or a complex continuum of three dimensions, and doubtless other forms of continua and discontinua.

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* This theorem may be stated as follows: If on an interval ab there is a set of intervals $[\sigma]$ such that (1) the points a and b are end points of intervals of the set $[\sigma]$; (2) every point of the interval ab lies within at least one interval of $[\sigma]$; then there is a finite subset $\sigma_1, \ldots, \sigma_k$ of the set $[\sigma]$ which satisfy (1) and (2) of the hypothesis.

The generality and high scientific interest of such a procedure are immediately evident. Heretofore the usual way has been to set up a system of assumptions sufficient and suitable for the real domain. So long a time elapses between the axiom stage and that where complex quantities are introduced that it does not occur to the student to ask whether the introduction of the new elements calls for a revision of the postulates or not. For instance, we assume that on any straight line we may find two distinct points whose distances from a given point are congruent to the distance of any two distinct points AB.* What happens in the imaginary domain when AB is null?

All such mishaps are avoided by the method adopted by our authors. There is nothing said which must later be unsaid, no assumption made which must later be discarded. At the beginning of each chapter we learn exactly what assumptions are needed for all of the theorems in that chapter; one or another assumption may not be needed, none is ever contradicted.

Such a proceeding is certainly attractive. With a minimum of assumptions, we have a maximum of generality. The complex domain is not a spirit world in which dwells the ghost that accompanies Steiner's quartic surface, but a larger ambient in which the real is immersed. But in mathematics, as everywhere else, the law of compensation holds. One can not obtain great results except at a great price, and the authors have paid a great price for the advantages obtained: they have sacrificed all that is specially characteristic of the real domain. Opinions will differ on the wisdom of such a bargain; in our judgment they have secured their gain at too great cost. Involutions appear on page 102, but from cover to cover no word as to the distinction between elliptic and hyperbolic. From page 109 onward the conic is continually in evidence, the words ellipse, hyperbola, parabola never occur. There is no way of knowing whether a coplanar line and conic intersect or not. Scientifically all this is a trivial matter, in higher mathematics the complex is on all fours with the real; but didactically it is momentous. The geometry with which we begin life, the geometry of intuition, the geometry of Euclid, is the geometry of the real continuum; it takes years of arduous training before the student can really think in any other terms, so long, that is, as he is doing geometry, and not merely formal algebra. Is it wise so far to discard psychology and historical tradition in founding projective geometry?

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* Hilbert's axiom, IV, 1.

The loss of the real domain is most keenly felt in the exercises, as there is no possibility of fruitful application of the methods developed to metrical theorems. Who that has taught projective geometry does not know what a sense of satisfaction comes over a class when they find a metrical theorem coming out of a projective one? They feel as if something familiar and concrete, almost practical, were emerging from that which had previously been abstract and remote. The student will be interested in the theorem (page 119) that if the vertices of two triangles lie on a conic, their sides touch another conic, and conversely. But he will be much more interested to find that if a triangle be circumscribed to a parabola, the circumscribed circle of the triangle passes through the focus of the parabola and, hence, that the circles circumscribing the four triangles formed by four lines are concurrent. A wide-awake student will be struck with the generality of Desargues' theorem (page 127), or its dual, whereby the pairs of tangents from any point to conics touching four lines, form pairs of an involution. Still more will he be pleased when he sees that the whole theory of confocal conics is deducible from this, or learns the Gauss-Bodenmiller theorem whereby the circles on the diagonals of a complete quadrilateral as diameters are coaxal. All of the best recent text books of projective geometry, Reye, Cremona, Russell, Enriques, Böger, agree on this didactic principle; even our authors go so far as to say (page iv) that the teacher who wishes for metrical applications may pass directly from Chapter VIII to the second volume. But the student who has mastered 235 pages of this volume will stand far less in need of metrical applications than his less fortunate brother who is at the beginning.

One other ground for postponing continuity and the real domain is offered by the authors (page iii) where they suggest that the study is too difficult and delicate for the beginner. "It will be found that the theorems selected on this basis of logical simplicity, are also elementary in the sense of being easily comprehended, and often used." Very well, how do we prove that a straight line and a conic in the same plane must intersect? We need proposition K_2, which is made to depend on K_n (page 255) proved at the end of Chapter IX: "*If* $a_x^k, a_x^l, \cdots$ *are a finite number of binary homogeneous forms, whose coefficients are proper in a space S which satisfies assumptions AEP, there exists a space S′, of which S is a sub-space, in the number system of which each of these forms is a product of linear factors.*"

We are not familiar with any treatment of continuity (not even the Heine-Borel theorem) whose essence seems to us so difficult to grasp as this. Moreover, we think that the question of the difficulty of the principle of continuity is a little beside the point. Students who take up the book before us are, presumably, just at that stage where a sound grasp of this principle is vital to further progress. We might well write over the doors of our graduate class-rooms for mathematics, a paraphrase of the motto which Plato is reputed to have placed over the door of the Academy

"Let no one ignorant of continuity presume to enter here."

We must not, of course, object if someone write below the equally famous

"Lasciate ogni speranza, voi che entrate."

No, we differ fundamentally from the present authors. We believe that the starting point of geometry, scientifically and didactically, should be the real continuum. So Enriques has started in the text-book already quoted. The complex domain rests on a convenient fiction whereby we employ the terminology of n dimensions to the study of $2n$ dimensions. A thorough study of continuity, which can be admirably given in projective geometry by starting from the idea of separation, is of primary importance to every graduate mathematical student, and should on no account be postponed.

There is a second general feature of the present book on which we dwell with unreserved satisfaction, the introduction of both algebraic and geometric methods. In the past there has been a very marked cleavage between two different schools of projective geometers. On the one hand we have the disciples of Chasles, Steiner, and von Staudt, purists, who treat an algebraic proof as an unseemly intrusion into the beautiful mansion of geometry. Their position was well expressed by the late Mr. Lachlan (Preface to his Modern Pure Geometry):

"In fact it might well be taken as an axiom, based on experience, that every geometric theorem admits of a simple and direct proof, based on the principles of pure geometry." Over against these we have the school of formalists, now happily dwindling, who burn incense to syzygies and symbolic identities, regardless of how much or little geometric interest they may have. The advanced student may freely enroll himself in either of these schools, if he so choose; the beginner should have both the algebraic and the geometric point of view continually before him. We are not familiar with any other book on projective geometry (not based on metrical assumptions) which has attempted to meet this important need. We only regret that the authors have not found it possible to introduce algebraic methods earlier into their scheme, and to carry the geometric and algebraic treatment of conics, collineations, etc., forward together. Let us mention also in this connection that the book has a greater wealth of suitable exercises for the student than any other with which we are familiar, with the single exception of Russell's Pure Geometry.

One or two more general remarks. The style is, for the most part, fresh and vigorous. Even Jove will nod. On page 36 the authors take an absolutely indecent pleasure in projecting points and lines *onto* planes; on page 224 we learn that AA' *is* a conjugate pair. The general effect is clear and agreeable. The figures, on the other hand seem to us occasionally messy and unattractive. Such a figure as No. 19—two quadrangles whose corresponding pairs of sides are concurrent on a given line—suffers not a little by comparison with the corresponding figure No. 15 in Reye's Geometrie der Lage. Otherwise, the book is singularly attractive in appearance, and free from typographical errors.

Our book starts off with an introductory chapter dealing with general principles concerning the use of undefined elements, unproved propositions, etc. Simple examples are given of axiom systems which are seen to be consistent and independent. The authors know very well what they are about; and the whole discussion is luminous. Unfortunately the chapter is marred by a slip, the worst in the book. In attempting to show how the point, line, and plane at infinity come in naturally to complete our geometry, the usual cartesian coördinates are mentioned. We then learn (page 9) "If the numbers are real numbers, we are dealing with the ordinary "real" space; if they are complex numbers, we are dealing with the ordinary "complex" space of three dimensions. The following discussion applies to either case." Then, on the following page, "Now the points (x, y, z) with the exception of $(0, 0, 0)$ may also be denoted by the direction cosines of the line joining the point to the origin of coordinates, and by the distance of the point from the origin, say by $(l, m, n, 1/d)$." This statement is neither grammatically correct nor mathematically accurate, for there are ∞^2 points for which $d = 0$. A reader finding such a mistake at the outset might well expect a looseness of statement throughout, and be unfairly prejudiced against the book in consequence.

In Chapter I we make a fair start into the theorems of alignment, and the principle of duality. The fundamental objects are *points* and classes of points called *lines*. We have the necessary assumptions and existence theorems, the plane is built up on a triangular frame, the space on a tetrahedral one. It is a little hard at first to become used to the idea of a point being *on* a plane, while the plane is, at the same time, *on* the point. The use of this locution (for which the authors give the credit to Professor Morley) is amply justified however, for it enables them to make the best statement of the principle of duality with which we are familiar. The chapter ends with an excellent system of assumptions for the projective geometry of n dimensions,—or rather should end there. Actually the last thing is this exercise: "State the assumptions of extension by which to replace E_n and E_n' for a space of an infinite number of dimensions. Make use of transfinite numbers." We are told in a note that exercises such as this, which are marked with an asterisk, are of a more advanced or difficult character. We are glad to note very few such exercises in the book.

Chapter II continues the same order of ideas as are introduced in Chapter I, and is largely devoted to configurations. Desargues's theorem on perspective triangles is, of course, fundamental for this. A "configuration," by the way, is an assemblage of points, lines, and planes, wherein each element of one sort is on a constant number of elements of each of the other sorts. These numbers may be arranged in a square matrix, which is taken to symbolize the configuration. We have an impression that many of the results of this chapter, which would be fundamental in the study of finite collineation groups may appear a little dull to the beginner, but we speak here from personal prejudice rather than from settled conviction. On page 45 is a little axiom which may strike the reader as suprising. "*The diagonal points of a complete quadrangle are not collinear.*" One has an instinctive idea that such a theorem is capable of proof by means of the other assumptions, but nothing is less reliable than our instinctive ideas on such matters. A rigorous proof may be found, based on the concept of separation.* A good part of the present book is independent of this assumption.

The third chapter is given to the study of projectivities. The authors have here cut loose from the von Staudt school where the quadrangle theorem is immediately followed by the study of harmonic separation. They, like Cremona and Böger, but unlike Reye and Enriques, define as projective two one-dimensional fundamental forms which may be connected by a finite number of projections and sections. The von Staudt definition whereby projectivity depends upon one to one correspondence, with a correspondence of harmonic elements is equivalent to this in the real domain, but not in the complex one (as the authors very well know), and so finds no place. So too, in two dimensions they distinguish between a collineation between two planes, a one to one correspondence point to point, and point range to point range, and a projective collineation, where to the above is added the requirement that corresponding point ranges shall be projective. Once more in the real domain collineation and projective collineation are identical, not in the complex one. The classic example is the transformation (which the authors would call a non-projective collineation but is more commonly known as an anticollineation), where each point is carried into its conjugate imaginary. Another commendable feature of the present chapter is that after a short discussion of the abstract concept of a group, the authors take up with some care the group of one-dimensional projectivities. It is not a little curious that previous writers of text-books on projective geometry have so calmly neglected this important topic.

Chapter IV is a natural continuation of Chapter III and opens with a discussion of harmonic sets. A new and important concept is introduced in the idea of *nets of rationality*. The assemblage of all points collinear with three given collinear points, and obtainable from them by a finite number of harmonic constructions, is called the net of rationality of the first three. We have corresponding point assemblages in two and three dimensions, the discussion of the latter (pages 89–93) seems to us somewhat unnecessarily difficult. The underlying idea is important, and is an excellent innovation in the present book. This same chapter leads us also to the fundamental theorem of projective geometry. Suppose that we have two projective one-dimensional fundamental forms (point ranges, linear pencils, etc.) with three elements self-corresponding, every element of the net of rationality of those three is self-corresponding, as may be immediately shown. The fundamental theorem (here assumption P) tells us that *every* element is self-corresponding. There are proofs in plenty of this theorem (Vahlen, Enriques, etc.) based on axioms of continuity, but the present authors have no such axiom in this first volume, and they have evolved beyond the intuitive proofs which satisfy Reye and Böger, so they adopt the natural alternative of taking it as an unproved assumption. The chapter closes with a short account of involutions, and a classification of two-dimensional collineations, based upon the discussion of the fixed elements.

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* Conf. Enriques, "Geometria Proiettiva," p. 59.

Chapter V introduces the point conic, the line conic, and the conic "überhaupt" which is conceived as a one-parameter assemblage of point-line unions. The treatment follows somewhat closely that of Reye. Conics are first defined by means of projective pencils and ranges; Pascal and Brianchon then enter hand in hand, and furnish the basis for the study of the polar system. It is interesting to compare this, the Steiner method, with that of von Staudt which consists in beginning with the polar system, and defining the conic by means of the points which lie on their polar lines. The question as to which method is finally the best is not capable of categorical answer. The Steiner method gives at once a way of constructing as many points of a conic as we please. Moreover the conic is probably the most interesting figure of all to the beginner, and Steiner leads us to it in the shortest time. Our authors reach the conic on page 108, Reye, who does not bother his head about axioms, brings conics in on page 66; in Enriques, the best book on the von Staudt plan, 202 long pages must pass before conics are reached. On the other hand by the von Staudt definition conic and polar system come in together, so that the polar properties are an immediate consequence of the definition; the same is true of the equation of the conic, which may be written immediately from that of the correlation, and the procedure works equally easily in any number of dimensions. However, we digress. The comparison of these two methods, interesting as it may be in itself, has no particular bearing on the present book, at least in its present order. The authors pass over another one hundred forty pages before taking up the question of the double points of an involution. Consequently, had they chosen the von Staudt definition, there would have been absolutely no way of telling whether a conic had any points or not. The latter part of the chapter goes to Desargues' involution theorem, and to pairs of conics. There is a certain vagueness in the statement of the former which should be remedied. We read (page 129) "*Any line, not through a vertex of the determining quadrangle, is met by the conics of a pencil of Type I in the pairs of an involution.*" Now as this sentence stands, it says emphatically that every line in the plane of a conic will meet that conic at least once, an admission which the authors have no wish to make at present. What they mean is that those conics of the pencil which meet the line do so in pairs of an involution.

An important change comes over the face of the book with Chapter VI, where the algebraic treatment begins. We start with two single-valued operations on pairs of the points of a line called *addition* and *multiplication*. These are shown to be commutative, distributive, and associative. This premised, let us turn to arithmetic. The element here is called a *number*. It is not requisite that a number should be a class of classes, or a recognizable graphical symbol; what is important is that the assemblage of numbers should be subject to two operations $\oplus$ and $\odot$ which follow the five laws mentioned above. The content of the class is quite immaterial, the essential thing is the nature of the laws of association of the members of the class, or of their symbolic representations on paper. Suppose, now, that every point on a line, except a poor scape-goat called P_∞, be represented symbolically by a finite number of pen-strokes. If we take x and y to stand for two such representations we may replace the symbol for the sum of the corresponding points by $x + y$ and that for the product of the points by xy.

This we believe to be the essence of the abstract isomorphism of point range and number field developed in the present chapter. One or two remarks may not be amiss. To begin with it is extremely abstract. The student learns at school that a point is that which has position and no magnitude. When in more advanced work he learns that the meaning of the word point is a variable rather than a constant, and in his projective geometry, as here studied, he finds no one cares exactly what a point may be, he will be rudely shaken. When he is told in the present chapter that the same vagueness hangs about the meaning of the word number, he is lucky if he retain any faith at all in God or man. It is probable therefore, that the student will have to go very slowly over all this ground, until the new and very strange ideas are so thoroughly familiar, that he can restate them in his own words; such at any rate was the case with the reviewer. We wonder, secondly, whether there is not another unproved assumption lurking somewhere in the idea that every point on a line may be represented on paper by some recognizable symbol. But here we see the Burali-Forti paradox lurking in ambush, and hasten on to safer ground.

When the correspondence of points on a line and numbers has been established, the whole algebraic apparatus may be set up in short order. Every point on the line has a coordinate, even P_∞ is welcomed by using homogeneous coordinates. The value of the cross ratio $ABCD$ is defined as x when

$$ABCD \barwedge P_\infty P_0 P_1 P_x.$$

A linear transformation of the coordinates will give a projective transformation of the line. These algebraic relations are easily extended to the plane (Chapter VII) and to space. Starting with two fundamental points in the plane, every point thereof not collinear with them has two definite coordinates, dependent on the cross ratios of quartets of lines through the fundamental points; a corresponding, though more elaborate, scaffolding will give three coordinates for points in space. We feel that the handling of this part of the subject is somewhat lacking in grace. Non-homogeneous point coordinates are never subsequently used, and might as well be omitted. The treatment of homogeneous coordinates seems less elegant than that given long ago in Pasch's "Neuere Geometrie."

We are not yet, by any means, through with one-dimensional projectivities. They return in force in Chapter VIII. We learn that projectivities with common fixed points are commutative. Then comes an elaborate discussion of projectivities and involutions on a conic, including the important theorem (page 220) "The group of projective collineations in a plane leaving a non-degenerate conic invariant is simply isomorphic with the general projective group of a line." The whole chapter is full of interesting and valuable material, we only regret that the algebraic aspect of projective transformations is not brought into closer relation with the geometric discussion.

In Chapter IX we reach two new and important ideas, the degree of a geometrical problem and invariance. The authors acknowledge (page 237) considerable indebtedness to Castelnuovo for the treatment of the first of these topics. Whatever the source, the result is admirable.

But now if we are to do any problems except those of the first degree, we must have some circumstances under which we are sure that a line will intersect a conic. The authors take the fatal plunge in proposition K_2, "*If any finite number of involutions are given in a space S satisfying assumptions AEP, there exist a space S′, of which S is a sub-space, such that all the given involutions have double points in S′.*"

This is a special case of proposition K_n already discussed. We fear that some students may find the proof of the latter difficult to follow; if they do not, they are more fortunate than the reviewer. We have already dwelt sufficiently upon the general aspects of this theorem, we therefore drop it with the remark that it gives us no light whatever on the question as to whether a conic and line intersect in S or not.

A third and last discussion of two dimensional collineations appears in Chapter X with a complete classification by means of elementary divisors. It is not made perfectly clear why distinct roots of the characteristic equation might not give the same double point in the collineation, although the proof is very easy. Correlations come in besides collineations, and receive an adequate algebraic treatment. There is a careful discussion of two-dimensional polar systems, and the chapter ends with a complete classification of pairs of conics. Let us repeat for the last time our regret that this clear and satisfactory algebraic handling is so far removed from the corresponding geometric one.

The concluding Chapter XI deals with systems of lines. Here we are frankly disappointed, except with the first and last sixths of the work. After a good discussion of the regulus we have (page 304):

"The lines joining corresponding points on two projective conics in different planes form a regulus, provided the conics determine the same involution I of conjugate points on the line of intersection l of the two planes, and provided the collineation between the two planes determined by the correspondence of the conics transforms l into itself by a projectivity to which I belongs."

The various "provideds" here mean that the two conics have two common points in the great space S', and these correspond to themselves in the projectivity. But like many a man who does not care to reflect on his future state, the authors prefer "One world at a time" and only use S' when they are driven to. Is it worth while to give at all a theorem whose statement is so long, and whose proof, with two corollaries covers six pages of text? The remainder of the geometrical part of the chapter hangs on the following definition (page 311):

"If two lines are coplanar, the lines of the flat pencil connecting them both are said to be *linearly dependent* on them. If two lines are skew, the only lines linearly dependent on them are themselves. On three skew lines are *linearly dependent* all the lines of the regulus of which they are rulers. If $l_1, l_2, \cdots, l_n$ are any number of lines and $m_1, \cdots, m_k$ are such lines that m_1 is linearly dependent on two or three of the lines $l_1, l_2, \cdots, l_n$, m_2 is linearly dependent on two or three of the lines $l_1, l_2, \cdots, l_n, m_1$, and so on, m_k being linearly dependent on two or three of the lines $l_1, l_2, \cdots, l_n, m_1, m_2, \cdots, m_{k-1}$, then m_k is said to be *linearly dependent* on the lines $l_1, l_2, \cdots, l_n$. A set of n lines no one of which is linearly dependent on the others are said to be *linearly independent*."

What the authors here mean is that lines shall be defined as linearly dependent when there is linear dependence among their Plücker coordinates. The reader, however, in facing this definition, is in the position of Pascal, who, in one of his inimitable "Lettres à un Provincial," thus speaks of a theological term, used by his adversaries: "Je fixai ce môt dans ma mémoire, car ça ne réprésenta rien à mon intelligence." The whole treatment of linear complexes and congruences is built on this basis (instead of on the null system which enters so naturally under correlations).

We have criticised this book freely; in our opinion no teacher should use it who is not sure enough of his ground to be willing to disagree with the authors on many points. But let no one criticise who is not able to recognise therein a fine piece of American scholarship.

JULIAN LOWELL COOLIDGE.

SHORTER NOTICES.

Diophantus of Alexandria. A Study in the History of Greek Algebra. By Sir THOMAS L. HEATH. Second Edition. Cambridge, University Press, 1910. vi + 387 pp.

THE edition of 1885 has been modified to such an extent that the present edition is in effect a new work. Most of the introductory chapters have been entirely rewritten. The account of the Arithmetica is fuller and a more literal translation of the original; the critical text by Tannery (Teubner, 1893, 1895) has been drawn upon to remove the former obscurity of certain passages.

The first edition paid little attention to the very important annotations by Fermat upon his copy of Bachet's edition (1621) of Diophantus. This deficiency has been remedied in the present work, which gives also much material selected from the correspondence of Fermat (Oeuvres, Paris, 1891–1896). Various investigations by Euler relating more or less closely to the problems discussed in the Arithmetica are given either as footnotes or in the final chapter, which devotes 52 pages to solutions by Euler. These extensive additions enable the reader to see in a concrete manner the influence which the work of Diophantus has had upon the development of the theory of numbers. Diophantus originated the theory of indeterminate equations; but in their solution he imposed the same limitations as in the case of determinate equations, requiring positive rational solutions. The restriction to integral solutions was made only in special cases, so that the present day designation of Diophantine equations is a misnomer.

It is stated on page 107 that "it is not probable that Diophantus was aware that prime numbers of the form $4n + 1$ and numbers arising from the multiplication of such numbers are the only classes of numbers which are always the sum of two squares." A factor 2^h may also be present.

On page 108 Heath states that we must credit Diophantus with the knowledge that no number of the form $8n + 7$ can be the sum of three squares. This is an illogical inference from the statement of Diophantus to the effect that no number of the form $3a + 1$, where $a = 8k + 2$, is the sum of three squares;

Diophantus makes no statement whatever about numbers of the forms $24n + 15$, $24n + 23$.

The limit to the numbers for which Bachet verified that each is the sum of at most four squares is stated to be 120 on page 110 and to be 325 on page 188.

On page 293, line 13 from bottom, Fermat is quoted as saying, "But both these squares can be shown to be smaller than the squares $\cdots$," whereas the translation should be, "But the sum of these two squares can be shown to be smaller than that of the first two $\cdots$" (Oeuvres, 3, page 272). The former reading would change the nature of the proof, and is moreover not in accord with the next sentences of the quotation. The same incorrect version occurs in the first footnote on page 295, in which Heath points out that Zeuthen changed (in another manner) the reasoning of Fermat.

While in the first edition the numbering of the problems in each book of the Arithmetica was in accord with the usage of other writers, this is unfortunately not the case with the present edition. Sections which give lemmas needed in subsequent sections are left unnumbered (and so in effect fall into the irrelevant earlier sections). While it is certainly desirable to have the heading which indicates this fact that the section is a lemma to a later section, the omission of a number for the section is a defect. But of course the chief objection is the lack of uniformity with established usage, which lessens the value of the work as a reference book. In histories of mathematics and in texts on the theory of numbers there occur references to the comments by Bachet or Fermat on Diophantus IV, 20, for example. Changes in the standard numbering are most unfortunate.

It was only after a careful examination that the reviewer was able to locate these few defects. It is obvious that the book is the outcome of a great amount of careful investigation on the part of an expert on the earlier and later history of mathematics. The result is an attractive and trustworthy account of the work of Diophantus and a comprehensive analysis of its historical setting and its influence upon the subsequent development of algebra and the theory of numbers.

L. E. DICKSON.

Notice sur le Système des six Coordonnées homogènes d'une Droite et sur les Éléments de la Théorie des Complexes linéaires. Par A. SÉFÉRIAN. Lausanne, A. Dénéréaz-Spengler et Cie., 1910. 79 pp.

To quote from the preface: "At the end of the introduction to his work Statique graphique, Professor Mayor supposes the reader familiar with line coordinates and the theory of the linear line complex. The present work develops this theory and has for aim the preparation of the reader to read profitably Professor Mayor's book."

A point (x, y, z) and the direction cosines (X, Y, Z) of a line through it uniquely fix the latter. The projections (L, M, N) of the moments $L=yZ-zY$, etc., upon the coordinate planes furnish three further equations, from which x, y, z can be replaced by L, M, N. The six larger letters are the coordinates under discussion. Various theorems are now developed regarding relations between vectors and their projections, all results being expressed in terms of the coordinates.

The moment of two vectors is derived; it vanishes when the lines defining the vectors intersect. Linear systems are considered, pencils, bundles, reguli, all of them in vector language and, in part, vector notation. The moment of two linear systems furnishes a natural introduction to the concept of the linear complex. The ideas developed are polarity, conjugate lines, condition for involution, and the invariance of the anharmonic ratio under linear point transformations and duality. Finally, various points of contact with Professor Mayor's book are pointed out. The work does not pretend to be a mathematical contribution, but to contain a mechanical interpretation of well-known ideas, without any of the purely geometric applications.

VIRGIL SNYDER.

Theoretical Mechanics. By PERCEY F. SMITH, and WILLIAM RAYMOND LONGLEY. Boston, Ginn and Company, 1910. 288 pp.

THIS is an attractive volume of 288 pages in a binding similar to the other mathematical texts edited by Professor Smith. The pages have a neat appearance, the size of type and spacing being excellent, and the numerous diagrams throughout the book could scarcely be improved.

From the preface one learns that the text is intended primarily for the mathematical student and a perusal of the volume confirms the statement. The analytic side of the subject is kept constantly in the foreground while the geometric or intuitional side is wholly in the background. Thus the mathematical student will find here an excellent field for applying the ideas which he gathered in the calculus. Integrals of various kinds and differential equations of certain types are abundantly in evidence.

The subject matter of the volume is almost exclusively dynamics and it would seem that "Analytical dynamics" would have been a more accurate title than "Theoretical mechanics." Of the thirteen chapters eight deal with the motion of a particle or a system of particles, one with the motion of a rigid body and two with work and energy. Thus with the exception of the first and last chapters the entire book is devoted to kinematics and dynamics.

The first chapter is devoted to the integrals arising in the computation of centers of gravity, moments of mass, and moments of inertia. The authors make the point that a student fresh from the study of the integral calculus can dispose of these integrals before the real study of mechanics is begun. From a purely mathematical point of view this is true, for one can evaluate a set of integrals quite independently of whether or not there is any physical concept underlying them. From a mechanical point of view, however, it is not true, for the underlying physical concept is the main point. It is rather awkward to open a book with a reference to future chapters, and "The student is asked to accept these formulas as *definitions*" has quite the flavor of an apology. If this first chapter were the last chapter in a text-book of calculus the necessity for this would be apparent, but to be compelled to apologize for a lack of mechanical ideas in a text-book on mechanics merely shows that the subject matter of the chapter should not be at the beginning of the book. It is in the wrong place.

The second chapter takes up the subject of the motion of a point along a straight line. In the third chapter, on the curvilinear motion of a point, the conception of vectors is introduced and a very good discussion is given. The fourth chapter introduces the conception of force. The first of Newton's laws of motion is stated, the second is described, while the third is reserved for the following chapter. The description of the second law is not accurate. It says "change in momentum is caused by forces acting upon the body," and again "Force is therefore the cause of acceleration." As a matter of fact, Newton's second law says nothing about causation. It merely defines what is to be understood by "force" and how it is measured.* Axioms I and II, page 100, were considered by Newton as corollaries to the second law and not as independent axioms. In the fifth chapter, entitled "Work, energy, impulse," one finds a short discussion of constrained motion which requires the statement of Newton's third law. This subject seems to belong rather to the preceding chapter. Had it been placed there the three laws of motion could have been brought together and their fundamental importance in mechanics properly emphasized.

Chapters six, seven, eight, and nine deal respectively with motion under constant forces, central forces, harmonic motion, and motion in a resisting medium. These chapters are well written. Chapter ten, on the potential and potential energy, is short but clear. One would rather expect chapter eleven, on systems of material particles, to contain an application of the potential, but such is not the case. The idea of the potential is developed but is left barren. The twelfth chapter, on dynamics of rigid bodies, contains a relatively small number of theorems and a relatively large number of problems. Indeed we find nine pages of text and six pages of problems.

Chapter thirteen, the last chapter in the book, contains a few theorems in statics. It is wholly inadequate. Its deficiency in this respect is well illustrated by the fact that so important a concept as a couple is nowhere mentioned. Perhaps the reason for this scant treatment of statics is the fact that statics does not lend itself to such beautiful mathematical developments as does the subject of dynamics. A mathematical student seeking exercise for his mathematical faculties and material for his mathematical ideas will find in this book a selection of topics well suited to him and a superabundance of problems. But a student of mechanics or physics who is interested in obtaining a concrete hold on mechanical principles

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*On this point Poincaré has made the following remark: "Quand on dit que la force est la cause d'un mouvement, on fait de la métaphysique, et cette définition, si l'on devait s'en contenter, serait absolument stérile. Pour qu'une définition puisse servir à quelque chose, il faut qu'elle nous apprenne a *mesurer* la force; cela suffit d'ailleurs, il n'est nullement nécessaire qu'elle nous apprenne ce que c'est que la force *en soi*, ni si elle est la cause ou l'effet du mouvement." *La Science et l'Hypothèse*, p. 120.

and is not interested in the mere mathematics of the subject will probably find the book unsuitable.

Some misprints have been noted, but they seem to be few. The footnote on page 99, "A finite periodic function of the time must have the form A sin $(bt+\nu)$ or A cos $(bt+\nu)$, where A, b and ν are constants," is a decidedly curious statement.

W. D. MACMILLAN.

Die partiellen Differentialgleichungen der mathematischen Physik. Nach Riemann's Vorlesungen in fünfter Auflage bearbeitet von HEINRICH WEBER. Erster Band. Braunschweig, Vieweg und Sohn, 1910. xviii+528 pp. Unbound, M. 12; bound, M. 13.60.

THE earlier editions of this work have won for themselves an important place in every mathematical library, where they are frequently consulted both as reference books and for more consecutive reading supplementary to courses on mathematical physics. They have thus become so well known that an extensive review of the volume before us would be superfluous. Professor J. S. Ames has ably reviewed the fourth edition in this BULLETIN (volume 8, page 81). The following comment will therefore be limited to a comparison of the fourth and fifth editions.

In the preface to the fifth edition, Professor Weber remarks that some of the developments of mathematical physics during the past decade have been too significant to leave unmentioned, and that although they have not been carried sufficiently far to permit of finished exposition in a text-book, they should receive some attention in the pages to follow. He speaks of integral equations and the notion of relativity as two of the more important contributions made to the science since his last edition in 1900, and promises an application of the first in the present volume. The failure to find any satisfactory fulfillment of this promise will be the reader's greatest disappointment. In its place there appears a brief exposition of Neumann's method of the arithmetic mean, with the old assumption of a convex surface, a condition made unnecessary by the use of integral equations. A suggestion is indeed given for the removal of this restriction, but it is not developed further, and appears inadequate. It is not even pointed out that an integral equation is being solved, whereas, as Professor Mason has shown in his New Haven Colloquium lectures, Neumann's method admits a beautiful treatment in the garb of integral equations. A general outline of the theory of integral equations, sufficient for the purposes in hand, and quite in keeping with the character of the book, could have been given in a half dozen pages, and would have abbreviated the solution of the electrostatic problem which follows. A reference to Plemelj's monographs on the theory and its application to physical problems (*Monatshefte für Mathematik und Physik*, volume 15) would have been well in place.

The new edition has been freed from several errors occurring in the previous one. Two of these should be mentioned. The conditions that a function may be represented by a Fourier integral were incorrectly stated (4th edition, page 40). These have been corrected, and alternative conditions added. An example follows to which none of the criteria apply. It may be noted that Pringsheim has recently given broad conditions covering Weber's example (*Mathematische Annalen*, volume 68, pages 405–06). The expression for the potential energy of a pair of magnets (pages 366–67) has been corrected. A few minor errors have crept in, but their number is gratifyingly small. The definition of continuity of a function at infinity (5th edition, page 6), which characterizes the function as continuous in any finite subregion and as bounded, is unusual, to say the least.

The section on the gamma function will prove a welcome addition to the younger reader, and a number of references have been added at various points. This, however, is one respect in which greater improvement might have been made, since there has appeared recently a considerable number of excellent monographs and texts which would well supplement the briefer expositions of our author. The following additions appearing in the new edition have received from the major part of a section to a section apiece: Fourier's integral for several variables; the transformation of a differential expression of second order from one set of oblique coordinates to another; the definition of Bessel's functions of higher order in terms of the preceding ones; the expression of arbitrary functions in terms of integrals containing Bessel's functions; the solution in series of the general differential equation of Bessel; the expression for the divergence of a vector field in curvilinear coordinates; the values of the zonal harmonics for the values 0 and 1 of the argument. A whole division (Abschnitt) has been added on the general electrostatic problem; its first section deals with the general properties of Newtonian potentials and would seem to belong in an earlier division where these functions were first considered. The three remaining sections are devoted to the method of the arithmetic mean and Beer's treatment of the equilibrium of a charge on a conductor, with the example of the ellipsoid.

In general the author has been conservative in making changes. More valuable additions might have been made in some cases. Linear partial differential equations of second order, for instance, receive a scant two pages, which give only the device of finding particular solutions in the form of exponential functions with linear exponents. Undoubtedly the systematic development of integral equations and their applications, of the principle of relativity, and in all likelihood, of Lebesgue's theory of integration, will make necessary at some future time a book treating the same subjects in radically new ways. But for the present, and for some time to come, Weber's book, with its origins in Riemann's lectures, will continue to be indispensable. The new edition will find its way into the more complete libraries, although for smaller collections the previous one will prove wellnigh as useful, if a prediction can be made on the basis of the first volume of the new edition.

O. D. KELLOGG.

NOTES.

THE concluding (October) number of volume 34 of the *American Journal of Mathematics* contains the following papers: "The involutional birational transformation of the plane, of order 17," by VIRGIL SNYDER; "On the problem of two fixed centers and certain of its generalizations," by A. M. HILTEBEITEL; "Abstract definitions of all the substitution groups whose degrees do not exceed seven," by G. A. MILLER; "The attraction of a homogeneous spherical segment," by G. GREENHILL.

THE Prince Jablonowski society of Leipzig announces the following prize problem:

"What is the expression of the C. Neumann theorem (*Abhandlungen d. k. S. Gesellschaft d. Wissenschaften,* 1909) in the theory of the Newtonian potential, applied to the oval shaped solid derived from the ellipsoid by the method of reciprocal radii? The society desires that this question be answered, if not in its most general form, at least for some special case, for example, when the given ellipsoid is one of revolution, and the center of the transformation (to be carried out by the method of reciprocal radii) is at the center of the ellipsoid."

The value of the prize is 1,500 marks. Competing memoirs should be sent to the secretary of the society under the usual conditions before October 31, 1913.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY. The following advanced courses in mathematics are given during the present academic year.—By Professor H. W. TYLER: History of the mathematical sciences and teaching of elementary mathematics, two hours.—By Professor D. P. BARTLETT: Theory of probability and method of least squares, twenty periods, second half-year.—By Professor F. S. WOODS: Partial differential equations, two hours, first half-year; Elements of the theory of functions of a complex variable, two hours, second half-year; Advanced geometry, two hours, second half-year.—By Professor F. H. BAILEY: Fourier's series, Laplace's coefficients, two hours.—By Professor E. B. WILSON: Statistical methods in theoretical physics, one hour; Hydrodynamics, twenty periods, first half-year; Discrete systems, two hours, first half-year; Continuous media, two hours, second half-year.—By Professors TYLER, WOODS, WILSON, GEORGE: Advanced calculus, two to four hours.

THE following courses in mathematics are announced at the German universities during the winter semester, 1911–1912:

UNIVERSITY OF BERLIN.—By Professor H. A. SCHWARZ: Elementary geometric derivation of the most important properties of conics, two hours; Synthetic geometry, four hours; Theory of analytic functions, I, four hours; Colloquium, two hours; Seminar, two hours.—By Professor G. FROBENIUS: Analytic geometry, four hours; Theory of numbers, four hours; Seminar, two hours.—By Professor F. SCHOTTKY: Higher algebraic curves, four hours; Theory of automorphic and polygonal functions, four hours; Seminar, two hours.—By PROFESSOR G. HETTNER: Theory of probabilities and of errors of observation, two hours.—By Professor J. KNOBLAUCH: Differential calculus, four hours; with exercises, one hour; Theory of elliptic functions, four hours.—By Professor R. LEHMANN-

FILHÉS: Integral calculus, four hours; Determinants, four hours. —By Professor I. SCHUR: Theory of algebraic equations, four hours; Theory of functions, II, four hours.

UNIVERSITY OF BONN.—By Professor E. STUDY: Introduction to analytic geometry, four hours; Differential geometry, II, two hours; Seminar, two hours.—By Professor F. LONDON: Synthetic geometry, two hours; Differential and integral calculus, II, four hours; Seminar, two hours.—By Professor F. HAUSDORFF: Introduction to the theory of functions, four hours; Fourier series and related developments, two hours; Seminar, two hours.—By Dr. J. O. MÜLLER: Algebraic equations (Galois' theory), three hours.—By Professor PH. FURTWÄNGLER: Selected chapters of technical mechanics, two hours.

UNIVERSITY OF BRESLAU.—By Professor R. STURM: Plane analytic geometry, four hours: Geometric relations, II, four hours; Seminar, two hours.—By Professor A. KNESER: Elliptic functions, four hours; Integral equations, two hours; Seminar, two hours.—By Professor E. SCHMIDT: Integral calculus, four hours; Theory of sets, two hours; Seminar, two hours.—By Dr. W. SCHNEE: Differential geometry, three hours; Selected chapters in the theory of numbers, two hours.

UNIVERSITY OF ERLANGEN.—By Professor P. A. GORDAN: Theory of numbers, four hours.—By Professor M. NOETHER: Plane analytic geometry, four hours; Theory of abelian functions, four hours; Descriptive geometry, four hours.

UNIVERSITY OF FREIBURG: By Professor S. STICKELBERGER: Theory of functions, four hours; Higher plane curves, three hours; Seminar, two hours.—By Professor L. HEFFTER: Analytic geometry of space, four hours; with exercises, one hour; Analytic mechanics of connected masses, three hours; Seminar, one hour.—By Professor O. BOLZA: Introduction to the theory of integral equations, two hours; Seminar, two hours.—By Professor A. LOEWY: Differential calculus, four hours; with exercises, one hour; Theory of algebraic equations, II, three hours; Seminar, two hours.—By Dr. N. SEITH: Projective geometry, two hours.

UNIVERSITY OF GIESSEN.—By Professor E. NETTO: Higher algebra, three hours; Analytic geometry of space, three hours;

Seminar, two hours.—By Professor L. SCHLESINGER: Differential and integral calculus, with exercises, five hours; Theory of the logarithmic potential, four hours; Seminar, two hours.—By Professor H. GRASSMANN: Analytic mechanics, II, four hours.

UNIVERSITY OF GÖTTINGEN.—By Professor F. KLEIN: Introduction to differential and integral calculus, II, four hours; Seminar, two hours.—By Professor D. HILBERT: Logical foundations of mathematics, one hour; Mechanics of continua, four hours; Seminar, two hours.—By Professor E. LANDAU: Analytical theory of numbers, four hours; Seminar, two hours.—By Professor C. RUNGE: Mechanics with exercises, six hours.—By Dr. E. WEYL: Determinants, two hours; Advanced theory of functions, four hours; Exercises in the calculus, two hours.—By Dr. O. TOEPLITZ: Introduction to the theory of functions, four hours; with exercises, two hours.—By Dr. A. HAAR: Curves and surfaces, four hours.—By Professor F. BERNSTEIN: Calculus of insurance, two hours; Mathematics of statistics, three hours.—By Professor W. VOIGT: Partial differential equations of mathematical physics, four hours.

UNIVERSITY OF GREIFSWALD.—By Professor F. ENGEL: Theory of groups of transformations, II, four hours; Differential and integral calculus, I, four hours; Analytic geometry of space, two hours; Seminar, two hours.—By T. K. VAHLEN: Theory of functions, four hours; Kinematics and mechanism, one hour; Seminar, two hours.—By Dr. W. BLASCHKE: Analytic mechanics, II, four hours; Descriptive geometry, with exercises, four hours.

UNIVERSITY OF HALLE.—By Professor G. CANTOR: Differential and integral calculus, I, five hours; Seminar, two hours.—By Professor A. WANGERIN: Integral calculus, with exercises, four hours; Applications of elliptic functions, two hours; spherical trigonometry and geography, two hours; Seminar, two hours.—By Professor A. GUTZMER: Ordinary differential equations, four hours; Analytic mechanics, four hours; Seminar, two hours.—By Professor V. EBERHARD: Analytic geometry of space, four hours; Colloquium, two hours.

UNIVERSITY OF JENA.—By Professor J. THOMAE: Applications of the infinitesimal calculus to geometry, five hours.—

By Professor L. HAUSSNER: Theory of numbers, four hours; Differential and integral calculus, II, five hours; Analytic geometry of space, four hours; Proseminar, two hours; Seminar, one hour.—By Professor G. FREGE: Riemann's theory of functions, four hours; Symbolic notation, one hour.—By Dr. M. WINKELMANN: Technical mechanics, four hours; with exercises, one hour; Vector analysis, one hour.

UNIVERSITY OF KIEL.—By Professor L. POCHHAMMER: Differential geometry, four hours; Partial differential equations, four hours; Seminar, two hours.—By Professor G. LANDSBERG: Integral calculus, four hours; Higher algebra, four hours; Seminar, two hours.—By Professor M. DEHN: Introduction to higher analysis, three hours; Analytic geometry of space, four hours; Exercises in applied mathematics, two hours.

UNIVERSITY OF KÖNIGSBERG.—By Professor W. F. MEYER: Integral calculus, three hours; with exercises, one hour; Introduction to higher geometry, three hours; Seminar, two hours. —By Professor A. SCHOENFLIES: Differential geometry of curves and surfaces, four hours; Seminar, two hours.—By Dr. A. KALUZA: Analytic geometry of space, four hours; with exercises, one hour; Determinants, two hours.—By Dr. W. BIEBERBACH: Partial differential equations, two hours; colloquium, one hour.

UNIVERSITY OF LEIPZIG.—By Professor K. ROHN: Analytic geometry of space, four hours; Application of the differential calculus to space curves and surfaces, four hours; Seminar, two hours.—By Professor O. HÖLDER: Higher algebra, particularly Galois's theory of equations, two hours; Elliptic functions, four hours; Seminar, two hours.—By Professor G. HERGLOTZ: Mechanics, five hours; Fourier series and definite integrals, two hours; Seminar, two hours.—By Professor P. KOEBE: Differential and integral calculus, five hours; with exercises, one hour; Potential theory and partial differential equations, two hours.—By Dr. R. KÖNIG: Linear differential equations in a complex field, two hours.

UNIVERSITY OF MARBURG.—By Professor K. HENSEL: Integral calculus, four hours; Algebra, four hours; Seminar, two hours.—By Professor E. NEUMANN: Analytic theory of differential equations, four hours; Proseminar, two hours; Seminar, two hours. —By Professor W. v. DALWIGK: Analytic mechanics, II, two hours; Advanced chapters from the theory of analytic functions, two hours; Perspective and photogrammetry, two hours; with exercises, three hours.—By Dr. E. HELLINGER: Analytic geometry of space, particularly of quadric surfaces, four hours; Selected chapters from the theory of functions of real variables, one hour; Exercises in the lower Seminar, one hour.

UNIVERSITY OF MUNICH.—By Professor F. LINDEMANN: Differential calculus, five hours; Theory of abelian functions, four hours; Mathematical foundation of insurance, two hours; Seminar, two hours.—By Professor A. VOSS: Plane analytic geometry, four hours; Mechanics, I, four hours; Seminar, two hours.—By Professor A. PRINGSHEIM: Elements of the theory of numbers, four hours; New methods and results in the theory of functions, four hours.—By Professor H. BRUNN: Elements of higher mathematics and foundations of descriptive geometry with exercises, four hours.—By Professor K. DOEHLEMANN: Descriptive geometry, I, five hours; with exercises, three hours; Synthetic geometry, five hours; with exercises, one hour.—By Dr. G. HARTOGS: Elementary geometry of two and three dimensions; Complement of algebraic analysis, one hour.

UNIVERSITY OF ROSTOCK.—By Professor O. STAUDE: Differential and integral calculus, four hours; Applications of the differential calculus to geometry, four hours; Seminar, two hours.

UNIVERSITY OF STRASSBURG.—By Professor H. WEBER: Differential and integral calculus, four hours; Partial differential equations of mathematical physics, two hours; Seminar, two hours.—By Professor F. Schur: Analytic geometry of the plane and of space, four hours; Selected chapters from the theory of surfaces, one hour; Seminar, two hours.—By Professor M. SIMON: Plane and spherical trigonometry, two hours; Noneuclidean geometry, two hours.—By Professor J. WELLSTEIN: Applications of the infinitesimal calculus, four hours; Determinants, two hours; Seminar, two hours.—By Professor P. EPSTEIN: Fourier series and related developments, two hours.

UNIVERSITY OF TÜBINGEN.—By Professor A. v. BRILL: Introduction to higher mathematics, four hours; Theory of algebraic curves, three hours; Seminar, two hours.—By Professor L. MAURER: Higher analysis, II, four hours; Higher algebra, three hours; Seminar, two hours.—By Professor O. PERRON: Elementary analysis, four hours; Descriptive geometry, three hours; with exercises, three hours; Seminar, two hours.

UNIVERSITY OF WÜRZBURG.—By Professor G. ROST: Analytic geometry of space, four hours; Plane analytic geometry, four hours; Spherical astronomy, II, two hours; Political arithmetic, two hours; Seminar, two hours; Introduction to higher mathematics (for students of insurance), two hours.—By Professor E. v. WEBER: Differential calculus, four hours; with exercises, two hours; Introduction to vector analysis, two hours; Seminar, two hours.—By Dr. E. HILB: Algebraic curves, four hours; Definite integrals, four hours; Seminar, two hours.

ON June 14 an academic prize was awarded by the philosophical faculty of the University of Göttingen to Mr. DUNHAM JACKSON, Hooper Fellow of Harvard University, for an essay entitled: "Ueber die Genauigkeit der Annäherung stetiger Funktionen durch ganze rationale Funktionen gegebenen Grades und trigonometrische Summen gegebener Ordnung."

THE Gauss monument on the Hohenhagen, near Dransfeld, was dedicated July 31, the memorial address being delivered by Professor W. VOIGT, of the University of Göttingen.

DURING the Easter vacation of 1912 an extensive course in mathematics and physics for advanced teachers will be held at Göttingen under the direction of Professor F. KLEIN.

PROFESSOR R. GANS, of the University of Tübingen, has been transferred to the University of Strassburg.

DR. R. KÖNIG has been appointed docent in mathematics at the University of Leipzig.

DR. H. v. SANDEN has been appointed docent in mathematics at the University of Göttingen.

DR. R. SCHIMMACK has been appointed docent in mathematical pedagogy at the University of Göttingen.

DR. L. SCHLEIERMACHER has been appointed docent in mathematics at the technical school of Darmstadt.

PROFESSOR E. FISCHER, of the technical school at Brünn, has accepted a professorship of mathematics at the University of Erlangen.

Dr. R. Fuchs, of the technical school at Berlin, has been promoted to an associate professorship of mathematics.

Professor H. v. Koch, of the technical school at Stockholm, has been appointed professor of mathematics at the University of Stockholm, as successor to Professor G. Mittag-Leffler, retired.

Dr. G. Majcen, of the University of Agram, has been promoted to a professorship of mathematics.

Dr. E. Salkowsky, of the technical school at Berlin, has been promoted to a professorship of mathematics.

Professor A. Schoenflies, of the University of Königsberg, has accepted the professorship of mathematics at the academy of social science at Frankfort.

Dr. A. N. Whitehead has been appointed lecturer in applied mathematics at University College, London.

Professor B. F. Yanney, of Mount Union College, Ohio, has accepted the professorship of mathematics at Wooster University.

Dr. H. E. Buchanan has been appointed professor of mathematics at the University of Tennessee.

Professor H. L. Rietz, of the University of Illinois, has been promoted to an associate professorship of mathematics. Dr. E. B. Stouffer has been appointed instructor in mathematics.

Mr. O. T. Geckeler has been appointed instructor in mathematics in the school of applied science of the Carnegie Technical Schools, Pittsburgh, Pa.

Dr. F. T. H'Doubler has been appointed instructor in mathematics at the University of Wisconsin.

Mr. A. B. Dunning has been appointed assistant professor of mathematics at Boston University.

Dr. E. G. Bill has been appointed instructor in mathematics at Purdue University.

Dr. S. Lefschetz, of Clark University, has been appointed instructor in mathematics in the University of Nebraska.

At Northwestern University Mr. C. H. Yeaton has been appointed instructors in mathematics.

Mr. L. C. Cox has been appointed instructor in mathematics at Pennsylvania State College.

Professor Franz Mertens, of the University of Vienna, has retired from active service.

Professor E. L. Hancock, of the Worcester Polytechnic Institute, died on October 1 at the age of thirty-eight years. Professor Hancock had been a member of the American Mathematical Society since 1902.

Professor J. Grünwald, of the German University at Prague, died July 1, at the age of 35 years.

Professor H. Schubert, of the Johanneum in Hamburg, and editor of the Sammlung Schubert, died July 20, at the age of 63 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Ahrens (W.). Gelehrten-Anekdoten. 2ter Teil. Berlin-Schöneberg, Sack, 1911. M. 2.40

Ambrecht (A.). Zum grossen Fermatschen Satz. Dresden, Kohler, 1911. 8vo. 4 pp. M. 0.50

Aussant–Carà (P.). Sulla discussione dei probleme riducibili al secondo grado, con applicazioni. Livorno, 1910. 8vo. 58 pp. M. 1.50

Bauch (B.). Studien zur Philosophie der exakten Wissenschaften. Heidelberg, 1911. M. 5.00

Bernstein (S. N.). The theory of probabilities. (Russian.) Charkow, 1911. 8vo. 254 pp.

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Book No.
L R·RY

A GENERALIZATION OF LINDELÖF'S THEOREMS ON THE CATENARY.

BY PROFESSOR OSKAR BOLZA.

THE object of the following note is to generalize two well known theorems on the catenary due to Lindelöf[*] by proving the following proposition:

Whenever the general integral of Euler's differential equation for the integral

$$J = \int f(x, y, y')dx \tag{1}$$

is of the form

$$y = \alpha\varphi\left(\frac{x - \beta}{\alpha}\right) \tag{2}$$

(*with* α, β *as constants of integration*), *the following two theorems hold:*

A) If A and A' are a pair of conjugate points (*in the wider sense*) *on an extremal for the integral* (1), *then the tangents to the extremal at A and A' meet at a point T of the x-axis, and vice versa.*[†]

B) The value of the integral J taken along the arc AA' of the extremal is equal to the value of J taken along the broken line ATA':[‡]

$$J(AA') = J(AT) + J(TA'). \tag{3}$$

The proof of the first theorem is almost immediate. For if

$$\mathfrak{E}_0: \quad y = \alpha_0\varphi\left(\frac{x - \beta_0}{\alpha_0}\right) \tag{4}$$

be any particular extremal of the family (2) and $A(x_1, y_1)$ one of its points, then the abscissa x_1' of the conjugate point A' satisfies the equation

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[*] Compare Lindelöf-Moigno, "Leçons sur le calcul des variations," pp. 209–213.

[†] Compare my "Vorlesungen über Variationsrechnung," p. 80.

[‡] L. Biianchi has recently generalized Lindelöf's second theorem from the integral $\int y\sqrt{1 + y'^2}dx$ to the more general integral $\int y^\nu\sqrt{1 + y'^2}dx$, *Rendiconti della R. Accademia dei Lincei, Classe di scienze fisiche, matematiche e naturali*, series 5, vol. 19 (1910), p. 705. The extremals for this integral are of the form (2), so that Bianchi's result is contained as a special case in our theorem B).

$$(5) \qquad u_1' - \frac{\varphi(u_1')}{\varphi'(u_1')} = u_1 - \frac{\varphi(u_1)}{\varphi'(u_1)},$$

where

$$u_1 = \frac{x_1 - \beta_0}{\alpha_0}, \qquad u_1' = \frac{x_1' - \beta_0}{\alpha_0}.$$

On the other hand, the abscissa of the point of intersection T of the tangent to $\mathfrak{E}$ at A with the x-axis is

$$(6) \qquad x_1 = \beta_0 + \alpha_0\left(u_1 - \frac{\varphi(u_1)}{\varphi'(u_1)}\right),$$

and the analogous quantity for the point A'

$$x_1' = \beta_0 + \alpha_0\left(u_1' - \frac{\varphi(u_1')}{\varphi'(u_1')}\right),$$

which, combined with (5), proves the first theorem.

In order to prove the second theorem we make use of a slight extension of Zermelo-Kneser's "envelop-theorem."*

Let

$$(7) \qquad y = Y(x, a)$$

be a one-parameter set of extremals for the general integral

$$J = \int f(x, y, y')dx,$$

and suppose there exist two curves $\mathfrak{K}_1$ and $\mathfrak{K}_2$ both tangent to all the extremals of the set (4) (two branches of the envelop of the set (4)). If $\tilde{x}_1(a)$ and $\tilde{x}_2(a)$ are the abscissas of the points of contact 1 and 2 of the extremal (a) with the curves $\mathfrak{K}_1$ and $\mathfrak{K}_2$ respectively, these two curves may be written in parameter representation

$$\mathfrak{K}_1: \qquad x = \tilde{x}_1(a), \ y = Y(\tilde{x}_1, a) \equiv \tilde{y}_1(a),$$

$$\mathfrak{K}_2: \qquad x = \tilde{x}_2(a), \ y = Y(\tilde{x}_2, a) \equiv \tilde{y}_2(a),$$

and

$$(8) \qquad Y'(\tilde{x}_1, a) = \frac{\tilde{y}_1'(a)}{\tilde{x}_1'(a)}, \qquad Y'(\tilde{x}_2, a) = \frac{\tilde{y}_2'(a)}{\tilde{x}_2'(a)},$$

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*Compare Kneser, *Mathematische Annalen*, vol. 50 (1898), p. 27.

whence it follows that

$$Y_a(\tilde{x}_1, a) = 0, \quad Y_a(\tilde{x}_2, a) = 0. \tag{9}$$

We now consider the integral J taken along the extremal (a) of the set (7) from the point 1 to the point 2. Its value is a function of a which we denote by $J(a)$, so that

$$J(a) = \int_{\tilde{x}_1}^{\tilde{x}_2} f(x, Y, Y')\, dx.$$

The derivative of this function can easily be computed by methods well known in the calculus of variations; if we take into account the equations (8) and (9), we obtain

$$J'(a) = f\left(\tilde{x}_2, \tilde{y}_2, \frac{\tilde{y}_2'}{\tilde{x}_2'}\right)\tilde{x}_2' - f\left(\tilde{x}_1, \tilde{y}_1, \frac{\tilde{y}_1'}{\tilde{x}_1'}\right)\tilde{x}_1'.$$

We integrate this equation with respect to a from a' to a'' and denote by $\mathfrak{E}'$ and $\mathfrak{E}''$ the extremals $a = a'$ and $a = a''$ of the set (7), by P', P'' the points of contact of $\mathfrak{K}_1$ with $\mathfrak{E}'$ and $\mathfrak{E}''$ respectively, by Q' and Q'' the points of contact of $\mathfrak{K}_2$ with $\mathfrak{E}'$ and $\mathfrak{E}''$ respectively. Then the result of the integration is

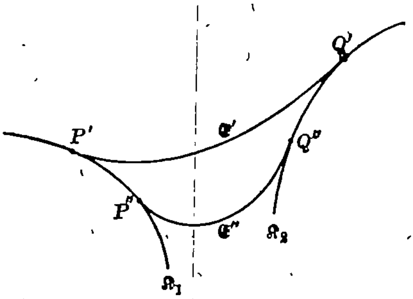

FIG. 1.

$$J_{\mathfrak{K}_1}(P'P'') + J_{\mathfrak{E}''}(P''Q'') = J_{\mathfrak{E}'}(P'Q') + J_{\mathfrak{K}_2}(Q'Q'').$$

But

$$J_{\mathfrak{K}_2}(Q'Q'') = -J_{\mathfrak{K}_2}(Q''Q').$$

Hence we obtain the desired extension of the envelop theorem in the form

$$J_{\mathfrak{E}'}(P'Q') = J_{\mathfrak{K}_1}(P'P'') + J_{\mathfrak{E}''}(P''Q'') + J_{\mathfrak{K}_2}(Q''Q'). \tag{10}$$

Our theorem B) can now easily be seen to be a special case of the general theorem just proved. For if the general integral of Euler's equation for the integral (1) is of the form (2), and if the tangents to the particular extremal (4) at A and A' meet at a point T of the x-axis, then the set of arcs directly homothetic with the arc AA' with respect to the center of similitude T are likewise extremals. For if c be the abscissa of the point T, these homothetic arcs are given by the equation

$$y = a\alpha_0\phi\left(\frac{x - (a\beta_0 + c(1 - a))}{a\alpha_0}\right), \tag{11}$$

with a as ratio of similitude. They are therefore contained in the form (2) and are consequently extremals for the integral (1).

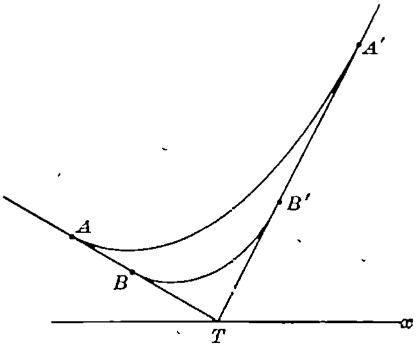

FIG. 2.

From the similitude of the arcs it follows that they are all touched at one of their extremities by the line TA, at the other by the line TA'. We may therefore apply to the set of extremals (11) and the two straight lines $\mathfrak{K}_1 = TA$ and $\mathfrak{K}_2 = TA'$ the above extension of the envelop theorem. Hence if BB' be one of the arcs in question for which $0 < a < 1$, then

$$J(AA') = J(AB) + J(BB') + J(B'A'),$$

and passing to the limit $a = 0$, we obtain the desired result

$$J(AA') = J(AT) + J(TA').$$

FREIBURG i. B.
July, 9, 1911.

A NOTE ON THE THEORY OF SUMMABLE INTEGRALS.

BY MR. S. CHAPMAN.

(Read before the American Mathematical Society, December 28, 1910.)

§ 1. *Introduction.*

Of recent years increasing use has been made of infinite series which do not converge; but whereas the mathematicians before the time of Abel and Cauchy used such series without proper examination of the validity of their use, modern mathematicians in general employ them only when the legitimacy of the work can be clearly demonstrated. The need so arising for theories to justify the application of ordinary methods and transformations to non-convergent series just as if they did converge has been fully recognized, and to meet it there have been published many well-known memoirs by Cesàro, Borel, Poincaré, LeRoy, and others. The theory is still being rapidly extended, and new important applications are constantly appearing. Among recent recent workers on the subject may be mentioned Bohr, Bromwich, Fejer, Hardy, C. N. Moore, and Riesz.

Parallel with the theory for infinite series there is a theory for infinite integrals, but the latter has not yet been developed so much as the former. Mr. Hardy* seems to have been the first to define a "summable" integral, and his paper was closely followed by one due to Dr. C. N. Moore,† in which some properties of summable integrals were proved. Subsequently Dr. Bromwich‡ wrote on the same subject, and more recently still the theory has been generalized by Mr. Hardy§ and myself.* In particular I have applied methods of "summation" to series and integrals which *converge*, with the object, not, of course, of finding their sum, but of gaining an insight into the nature of their convergence.

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* *Quar. Jour. of Mathematics*, vol. 35, p. 54.

† "On the introduction of convergence factors into summable series and summable integrals," *Trans. Amer. Math. Society*, vol. 8, p. 299.

‡ "On the limits of certain infinite series and integrals," *Math. Annalen*, vol. 65, p. 350.

§ G. H. Hardy, "Notes on some points in the integral calculus," *Messenger of Mathematics*, vol. 40. "Theorems connected with Maclaurin's test for the convergence of series," *Proc. Lond. Math. Soc.* G. H. Hardy and S. Chapman, "A general view of the theory of summability of series and integrals," *Quar. Jour. of Mathematics*, 1911.

§ 2. *Summable Integrals.*

Mr. Hardy's original definition of the summability of an integral was quite analogous to Cesàro's definition of the summability of a series by a first mean. Thus, if $f(x)$ is a function which is integrable in every finite interval lying in the interval $a \leqq x$, then the integral

$$\int_a^\infty f(x)dx$$

is said to be summable (C_1)† if the limit

$$\lim_{x \to \infty} \frac{1}{x} \int_a^x \int_a^\alpha f(\beta) d\beta d\alpha$$

exists; and the value of the limit is called the "sum" or "value" of the integral.

The extension of this definition is obvious, viz.,

The integral $\int_a^\infty f(x)dx$ is summable (C_r) if the limit

$$\lim_{x \to \infty} \frac{r!}{x^r} \int_a^x \int_a^{\alpha_1} \int_a^{\alpha_2} \cdots \int_a^{\alpha_r} f(\theta) d\theta d\alpha_r d\alpha_{r-1} \cdots d\alpha_1$$

exists; and as before the value of the limit defines the "sum" or "value" of the integral.

For many purposes, however, in the development of the general theory, the above multiple integral is more convenient when it is transformed into the expression

$$\int_a^x \left(1 - \frac{y}{x}\right)^r f(y) dy,$$

which is easily deduced by the theory of multiple integrals.

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* S. Chapman, "On non-integral orders of summability of series and integrals," *Proc. Lond. Math. Soc.*, 1911.

† The convenient notation of which this is a particular case was introduced by Mr. Hardy in a paper published in the *Proc. Lond. Math. Soc.*, series 2, vol. 4, p. 257.

In particular the latter form is valuable because it shows the analogy of the above method of summation with that introduced (for the case of series) by Dr. M. Riesz.*

Further, it suggests the idea of non-integral orders of summability, and is an eminently suitable form for proving theorems on summability in full generality and with extreme simplicity.

Dr. C. N. Moore, in the paper already quoted, proved that if an integral is convergent, it is summable (C_1) and its "sum" is the same as the ordinary value. This is a particular case of the more general theorem which I have established,† to the effect that "if an integral is summable (C_r), then it is summable $(C_{r'})$, with the same sum, provided $r' > r$, where r and r' may be any positive or negative numbers (including zero), greater than -1." This may be called the general condition of consistency.

In this paper, however, I shall restrict myself to integrals which are summable (C_r), where r is zero or a positive integer only. I propose to discuss the behaviour of $f(x)$ as $x \to \infty$, in such integrals. The main result is an extension (with a certain redundant condition removed) of a theorem by Dr. C. N. Moore in his paper on convergence factors.‡

§ 3. *The Limit of the Integrand of a Summable Integral.*

In the case of an infinite series, it is a necessary but insufficient condition for convergence that $\lim_{n\to\infty} a_n = 0$. More generally, for a series $\sum a_n$ to be summable (C_r), it is necessary (though insufficient) that $\lim_{n\to\infty} (a_n/n^r) = 0$; this has long been known to hold when r is a positive integer,§ and I have proved that it holds generally for any value of $r > -1$.

We should not expect the same result to hold in the case

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* *Comptes Rendus*, July, 1909. Dr. Riesz's definition of the sum of a non-convergent series is

$$\lim_{n\to\infty} \sum_{v=0}^{n-1} u_v \left(1-\frac{v}{n}\right)^r,$$

and evidently n may be integral or not.

† "On non-integral orders . . . ," *Proc. Lond. Math. Soc.*, 1911.

‡ The method also is an extension of the one there used, but the question itself was presented to me otherwise, by the corresponding theorem for series.

§ See, for instance, Bromwich, The Theory of Infinite Series, Ch. XI.

of integrals,* for even in the case of convergent integrals (when $r = 0$) we know that $f(x)$ need not approach 0, and in fact may have no finite upper limit. An example of this will be given almost immediately.

It is well-known, however, that if $\int_0^\infty f(x)dx$ converges, then the condition of *uniform* continuity of $f(x)$, for $x \geqq k$, where k is any constant, is sufficient to ensure that $f(x) \to 0$ as $x \to \infty$.

Simple continuity of itself implies uniform continuity over any finite closed range of the variable, but this is evidently not the case over an infinite range.

Dr. Moore, who considered only integrals summable (C_1), proved that if $f(x)$ is uniformly continuous for $x \geqq k > 0$, and if $\int_0^\infty f(x)dx$ is summable (C_1), then

$$\lim_{x \to \infty} f(x)/x = 0.$$

This is a particular case of the more general theorem which I proceed to establish; but first it must be mentioned that the restriction upon $f(x)$ of uniform continuity is unnecessarily stringent. It is sufficient that $f(x)/x$ should be uniformly continuous for $x \geqq k > 0$. As Dr. Moore himself proved in the memoir referred to, if $f(x)$ is uniformly continuous for $x \geqq k > 0$, then so also is $f(x)/x^r$, for any $r > 0$; but the converse does not hold.

For example, consider the function $f(x)$ defined thus: $f(x) = 0$ for

$$n + 1/n^2 \leqq x \leqq (n + 1) - 1/(n + 1)^2;$$

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* It may be interesting to point out that the method analogous to that which is successful in the case of series here fails completely. In proving the theorem for series, we use a formula for c_n in terms of $S_n{}^r$, viz.,

$$a = S_n^{(r)} - (r + 1) S_{n-1}^{(r)} + \frac{(r + 1)r}{1.2} S_{n-2}^{(r)} - \dots$$

to r or $n+1$ terms according as r is or is not a positive integer.

The integral analogue of this expression is easily seen to be

$$\int_0^x (x - z)^{-(r+\ )} \int_0^z f(t)(z - t)^r \, dt dz,$$

which is equal to

$$\int_0^x f(t)dt \int_t^x (z - t)^r (x - z)^{-(r+2)} dz = \int_0^x \frac{f(t)}{x - t} dt \int_0^1 \theta^r (1 - \theta)^{-(r+2)} d\theta.$$

The second integral converges for no value of r, and therefore the method is useless.

and $f[n \pm (1-\theta)/n^2] = \theta\sqrt{n}$ for $0 < \theta < 1$. $\int_0^\infty f(x)dx$ is convergent but $f(n) = \sqrt{n}$, so that $f(x)$ does not $\rightarrow 0$ as $x \rightarrow \infty$, nor has it even a finite upper limit. $f(x)$ is continuous but not uniformly so, for $x \geqq k > 0$. But since $\int_0^\infty f(x)dx$ is summable (C_0), by the general condition of consistency it is summable (C_1); also $f(x)/x \rightarrow 0$, which however is not deducible from Dr. Moore's theorem, since $f(x)$ is not *uniformly* continuous. On the other hand, $f(x)/x$ *is* uniformly continuous, and hence $f(x)/x \rightarrow 0$ does follow from the theorem as we shall prove it, with the unnecessary restriction removed.

§ 4.

THEOREM. *If $f(x)/x^r$ is uniformly continuous for $x \geqq k > 0$, and $\int_0^\infty f(x)dx$ is summable (C_r); then*

$$\lim_{x \rightarrow \infty} f(x)/x^r = 0.$$

I restrict myself to the case when r is a positive integer. There seems little doubt, however, that the theorem holds for all values of $n > -1$.

Let
$$L_r(x) = \int_0^\infty f(\alpha)\left(1 - \frac{\alpha}{x}\right)^r d\alpha.$$

By hypothesis, if L is the "sum" of the integral,

$$L_r(x) \rightarrow L \text{ as } x \rightarrow \infty.$$

Then, r being integral, we have

$$f(x) = \frac{1}{r!}\frac{d^{r+1}}{dx^{r+1}}\left[x^r L_r(x)\right], \tag{1}$$

which is immediately deducible from either form of $L_r(x)$. Since $f(x)/x^r$ is uniformly continuous for $x \geqq k > 0$, we can find a positive number δ, when any arbitrarily small positive constant ϵ is assigned, so that for $x \geqq k$,

$$\left|\frac{f(x+h)}{(x+h)^r} - \frac{f(x)}{x^r}\right| < \epsilon, \tag{2}$$

provided $0 < |h| < (r+1)\delta$.

But by the general mean value theorem of the differential calculus and equation (1) we have

$$(3)\quad \sum_{m=0}^{r+1}(-1)^m\binom{r+1}{m}(x+\overline{r+1-m\delta})^r L_r(x+\overline{r+1-m\delta})$$

$$= \delta^{r+1}\frac{d^{r+1}}{dx^{r+1}}[(x+\theta\delta)^r L_r(x+\theta\delta)]$$

$$= r!\,\delta^{r+1}f(x+\theta\delta)_x \qquad (0<\theta<r+1).$$

We now expand the expression (3) in the form of an integral polynomial in x. We get

$$x^r\sum_{m=0}^{r}(-1)^m\binom{r}{m}\{L_r(x+\overline{r+1-m\delta}) - L_r(x+\overline{r-m\delta})\}$$
$$+ x^{r-1}S,$$

where S contains only a finite number of terms (a number fixed when r is fixed) and, since $L_r(x)\to L$, has a finite upper limit K for all values of x and for all values of $\delta<1$, say. Further, we can find a constant x_0, depending on ϵ and δ, such that for $x'>x\geqq x_0$,

$$|L_r(x') - L_r(x)| < \epsilon\delta^{r+1}.$$

Hence for $x\geqq x_0$, we have

$$r!\,\delta^{r+1}f(x+\theta\delta) < 2^r\cdot x^r\cdot\epsilon\cdot\delta^{r+1} + x^{r-1}S,$$

or

$$\frac{f(x+\theta\delta)}{x^r} < \frac{2^r}{r!}\epsilon + \frac{K}{r!\,\delta^{r+1}x}.$$

Hence if $x_0 > 1/\delta^{r+1}\epsilon$,

$$\frac{f(x+\theta\delta)}{(x+\theta\delta)^r} < K'\epsilon,$$

where K' is independent of x and δ. Now θ is a function of x, and hence $x+\theta\delta$ does not necessarily assume all values along the real axis. But by equation (2) it follows that for all values of $x\geqq x_0 > 1/\delta^{r+1}\epsilon$

$$\frac{f(x)}{x^r} < K''\epsilon,$$

where K'' is a positive constant, and consequently $f(x)/x^r \rightarrow 0$ as $x \rightarrow \infty$, which was what we desired to prove.

In conclusion, we may remark that the theorem may be stated as one of pure integral calculus, without reference to the theory of summability of integrals. Putting $f(x) = \phi(x)x^r$, the theorem thus becomes:

If $\phi(x)$ *is uniformly continuous over the infinite interval* $x \geqq k > 0$, *then the convergence to a limit, as* $x \rightarrow \infty$, *of the integral*

$$\int_0^x \phi(\beta)\beta^r\left(1-\frac{\beta}{x}\right)^r d\beta$$

requires that $\phi(x)$ *shall* $\rightarrow 0$ *as* $x \rightarrow \infty$.

CAMBRIDGE,
ENGLAND.

IRREDUCIBLE HOMOGENEOUS LINEAR GROUPS OF ORDER p^m AND DEGREE p OR p^2.

BY PROFESSOR W. B. FITE.

(Read before the American Mathematical Society, February 25, 1911.)

No group all of whose non-invariant commutators give invariant commutators besides identity can be simply isomorphic with irreducible groups of different degrees. This category includes all groups of order p^m (p a prime) and classes one, two, and three. Moreover no group of order p^m can be simply isomorphic with irreducible groups of just two different degrees.*

A consideration of these facts gives rise to the query as to whether any group of order p^m can be simply isomorphic with irreducible groups of different degrees, and it is the purpose of this note to answer this question for certain special cases.

In the first place, if G is an irreducible group of order p^m and degree p, it cannot be simply isomorphic with an irreducible group of any other degree, since it contains an abelian subgroup of index † p, and since a group of order p^m with an abelian subgroup of index p^δ cannot be simply isomorphic with an irreducible group of degree greater than p^δ.*

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* BULLETIN, vol. 14 (1908), pp. 328, 329.

† *Transactions Amer. Math. Society*, vol. 7 (1906), p. 68. We shall have occasion to make use of the fact, established here, that in an irreducible group of order p^m and degree p, the substitutions commutative with a substitution that gives an invariant commutator besides identity form an abelian subgroup.

Suppose that G is an irreducible group of order p^m and degree p^2. If t is a substitution of G that corresponds to an invariant commutator of G' of order p, it is invariant in a subgroup G_1 of order p^{m-1} and G_1 must be reducible since t is invariant in it and is not a similarity substitution. We can so transform G as to exhibit G_1 in a completely reduced form with p irreducible components and with $G \equiv \{G_1, R\}$, where R replaces each variable of any component by the corresponding variable in the succeeding component.†

If G_1 contains a substitution s that has just p conjugates and is commutative with R, s must be the same in each of the components (each of degree p) of G_1. If s_i is that part of s that involves the variables of the ith component of G_1 then s_i is not invariant in its component; since if it were s would be invariant in G_1, and therefore invariant in G. Each component of G_1, as the ith one, being of degree p, contains an abelian subgroup of index p of which s_i is a part, and in the formation of G_1 no substitution of the abelian subgroup in one of these components can be associated with a substitution that is not in the corresponding abelian subgroup of some other component, since otherwise s would have more than p conjugates. Hence G_1 contains an abelian subgroup of index p, and G cannot be simply isomorphic with an irreducible group of any degree other than p^2.

Now t and R must be contained in a subgroup G_2 of order p^{m-1}, since otherwise G' would be of order p^2 and G could not be an irreducible group of degree p^2.‡ If then G_1 contains no substitution with just p conjugates that is commutative with R, G_2 can contain no invariant substitution that is not invariant in G. For every substitution of G_2 is of the form sR^α, where s is a substitution of G_1, and if sR^α is invariant in G_2, we must have $\alpha \equiv 0 \pmod{p}$, since s is commutative with t and R is not. Therefore sR^α is in G_1, and it is commutative with R. But G_1, by supposition, contains no substitution with just p conjugates that is commutative with R.

Hence G_2 is either irreducible or simply isomorphic with each of its p irreducible components.[*] In the latter case, since these components are each of degree p, G_2 contains an abelian subgroup of order p^{m-2} and G cannot be simply isomorphic with an irreducible group of any degree except p^2. In the former case, since G contains an irreducible subgroup of order p^{m-1} and degree p^2, it cannot be simply isomorphic with an irreducible group of any degree except p^2 and p^3, if we assume that no group of order p^{m_1} $(m_1 < m)$ can be simply isomorphic with irreducible groups of degrees p^2 and p^n respectively $(n > 2)$. But no group of order p^m can be simply isomorphic with irreducible groups of just two different degrees. Hence G cannot be simply isomorphic with an irreducible group of any degree except p^2. The assumption upon which this conclusion rests can easily be justified for small values of m. We have therefore proved the following

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* Loc. cit., vol. 8 (1907), p. 110.

† But the variables in the last component are not necessarily replaced by the corresponding ones of the first component.

‡ Loc. cit., vol. 7 (1906), p. 67, Theorem II.

THEOREM: *An irreducible group of order p^m and degree p^2 (p a prime) cannot be simply isomorphic with an irreducible group of any other degree.*

If we take into consideration the theorem just proved and the facts heretofore cited together with the fact that if G is simply isomorphic with an irreducible group of degree $p^{\frac{1}{2}(m-\alpha)}$, where p^α is the order of the central of G, it cannot be simply isomorphic with an irreducible group of any other degree,[†] we can readily verify that if a group of order p^m is simply isomorphic with irreducible groups of different degrees, then $m \geqq 12$.

Let G be an irreducible group of order p^m and degree p^2 that does not contain an abelian subgroup of index p^2. Then G_1 can contain no substitution that has just p conjugates and is commutative with R. Moreover the central of G' can contain only one subgroup of order p, since if t_1 and t_2 corresponded to independent operations of order p of the central of G', some substitution of the form $t_1t_2^\alpha$ would be commutative with R and would have just p conjugates in G_1. Hence the central of G' must be cyclic.[‡]

Any substitution of G that corresponds to an invariant operation of G' can give only invariant commutators; and if G' contains an invariant operation of order p^2, any corresponding substitution s of G has not more than p conjugates in G_1, since s^p is invariant in G_1. Moreover s is not invariant in G_1, since of its p irreducible components.[*] In the latter case, since these components are each of degree p, G_2 contains an abelian subgroup of order p^{m-2} and G cannot be simply isomorphic with an irreducible group of any degree except p^2. In the former case, since G contains an irreducible subgroup of order p^{m-1} and degree p^2, it cannot be simply isomorphic with an irreducible group of any degree except p^2 and p^3, if we assume that no group of order p^{m_1} $(m_1<m)$ can be simply isomorphic with irreducible groups of degrees p^2 and p^n respectively $(n>2)$. But no group of order p^m can be simply isomorphic with irreducible groups of just two different degrees. Hence G cannot be simply isomorphic with an irreducible group of any degree except p^2. The assumption upon which this conclusion rests can easily be justified for small values of m. We have therefore proved the following

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[*] Loc. cit., vol. 8 (1907), p. 107.

[†] Ibid., p. 109; and BULLETIN, 2d series, vol. 14 (1908), p. 328.

[‡] Burnside, Theory of Groups of Finite Order, pp. 73, 75.

THEOREM: *An irreducible group of order p^m and degree p^2 (p a prime) cannot be simply isomorphic with an irreducible group of any other degree.*

If we take into consideration the theorem just proved and the facts heretofore cited together with the fact that if G is simply isomorphic with an irreducible group of degree $p^{\frac{1}{2}(m-\alpha)}$, where p^α is the order of the central of G, it cannot be simply isomorphic with an irreducible group of any other degree,[†] we can readily verify that if a group of order p^m is simply isomorphic with irreducible groups of different degrees, then $m \geqq 12$.

Let G be an irreducible group of order p^m and degree p^2 that does not contain an abelian subgroup of index p^2. Then G_1 can contain no substitution that has just p conjugates and is commutative with R. Moreover the central of G' can contain only one subgroup of order p, since if t_1 and t_2 corresponded to independent operations of order p of the central of G', some substitution of the form $t_1t_2^\alpha$ would be commutative with R and would have just p conjugates in G_1. Hence the central of G' must be cyclic.[‡]

Any substitution of G that corresponds to an invariant operation of G' can give only invariant commutators; and if G' contains an invariant operation of order p^2, any corresponding substitution s of G has not more than p conjugates in G_1, since s^p is invariant in G_1. Moreover s is not invariant in G_1, since it is not commutative with R^p. Hence in each component of G_1, as the ith one, s_i is contained in an abelian subgroup of index p, and the subgroup of G_1 within which s is invariant is abelian and of index p^3 under G.

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[*] Loc. cit., vol. 8 (1907), p. 107.

[†] Ibid., p. 109; and BULLETIN, 2d series, vol. 14 (1908), p. 328.

[‡] Burnside, Theory of Groups of Finite Order, pp. 73, 75.

We can assume then that the central of G' is of order p, and hence that the central of no succeeding cogredient contains an operation of order greater than p. If G were of class 3, G'' would be of order* p^2 and G' of order p^3. But this is impossible for an irreducible group of degree p^2.† We can assume then that G is of class k, where $k \geqq 4$.

Suppose that t_2 is a substitution of G that corresponds to an invariant operation of G''. Then it must be in G_1,‡ and if $t_{2,i}$ were invariant in the ith component of G_1, $t_{2,j}$ $(j = 1, 2, \ldots, p)$ would be invariant in the jth component, since in $R^{-1}t_2R$ the ith component is the same (except for the names of the variables) as the $(i - 1)$th component of t_2, and since the commutator of R and t_2 is invariant in each component of G_1. Hence either t_2 is invariant in G_1, or $t_{2,i}$ is not invariant in the ith component and gives only invariant commutators. In the latter case the substitutions of the ith component of G_1 that are commutative with $t_{2,i}$ form an abelian group, and hence the substitutions of G_1 that are commutative with t_2 form an abelian group. Moreover this abelian group cannot be of index greater than p^2 under G_1, since t_2 cannot have more than p^2 conjugates under G_1. Hence G contains an abelian subgroup of index not greater than p^3.

We have now to consider the case in which t_2 is invariant in G_1. If under this supposition the central of G'' were of order greater than p, we could assume that every substitution of G that corresponded to an invariant operation of G'' is invariant in G_1. If then t_2 and s_2 were substitutions of G that corresponded to two independent operations of the central of G'', $t_2s_2^{\alpha}$ would, for a suitably chosen value of α, give only invariant commutators in G. But this is impossible. Hence we consider the central of G'' to be of order p.

If now G were of class 4 $(k = 4)$, G''' would be generated by two independent generators each of order p, and hence G_1 would be abelian. But this is impossible in an irreducible group of degree p^2.

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* Loc. cit., vol. 3 (1902), p. 351.

† Loc. cit., vol. 7 (1906), p. 67.

‡ Ibid., p. 62.

We now assume that an irreducible group of degree p^3 and order p^m contains an abelian subgroup of index less than p^k, if it is of class k, unless the central of each cogredient up to and including the $(k - j - 1)$th is of order p, and the substitutions of G that correspond to any operation of these centrals are invariant in G_1. In this excluded case, if t_{k-j} is a substitution of G that corresponds to an invariant operation of $G^{(k-j)}$, it is contained in G_1. If it is not invariant in G_1, the substitutions of G_1 that are commutative with it form an abelian subgroup of index not greater than p^{k-j} and G contains an abelian subgroup of index less than p^k. If t_{k-j} is invariant in G_1, it can easily be shown (as in the case $k = 4$) that we need only to consider the case in which the central of $G^{(k-j)}$ is of order p. Hence if our assumption holds for any value of $j (> 1)$, it holds for the next smaller value of j. But we have shown that it holds for $j = k - 2$.

We have assumed throughout that $k > 2$. If $k = 2$, G' is of order p^4 and contains an abelian subgroup of order p^{m-2}.*

We have proved therefore the

THEOREM: *If G is a group of order p^m and class k (>2) that is simply isomorphic with an irreducible group of degree p^3, it contains an abelian subgroup of index less than p^k. If $k = 2$, G contains an abelian subgroup of index p^2.*

COLUMBIA UNIVERSITY,
August, 1911.

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* Loc. cit., p. 67, and vol. 8 (1902), p. 343.

GRADUATE WORK IN MATHEMATICS IN UNIVERSITIES AND IN OTHER INSTITUTIONS OF LIKE GRADE IN THE UNITED STATES.*

1. *The Establishment of Advanced Instruction in the United States.*

FORTY years ago the bachelor's degree granted on the completion of a four years' course of a general character marked not merely the close of a young man's liberal education, but also, except in the case of some lawyers, ministers, and physicians, the end of all academic instruction of any kind. In particular, apart from a few exceptional cases, no advanced instruction in mathematics was anywhere provided beyond the usually rather meager ingredients—hardly more than analytic geometry and a little calculus—of this college course, which consisted mainly of prescribed studies. As an external sign of this state of affairs we note that the master's degree, where it existed, was conferred for reasons having very little to do with study, while the doctor's degree was practically non-existent.† The desire for higher education in America, which had been felt for many years by some of the leading minds of the country, had been able so far to achieve only momentary and sporadic success.

The most notable example of such a momentary success, so far as the study of mathematics is concerned, is to be found at Harvard during the fifties and early sixties, where, under the guidance of Benjamin Peirce, a band of young men devoted themselves successfully to the pursuit of higher mathematics.‡

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* *International Commission on the Teaching of Mathematics.* Committee XII. Chairman, MAXIME BÔCHER, Harvard University; D. R. CURTISS, Northwestern University; P. F. SMITH, Yale University; E. B. VAN VLECK, the University of Wisconsin.

The subcommittee reports here referred to are:

1. Courses of Instruction and the Master's Degree. Chairman, Professor Curtiss. Cf. BULLETIN, February, 1911.
2. Preparation for Research and the Doctor's Degree. Chairman, Professor Smith. Cf. BULLETIN, March, 1911.
3. Preparation of Instructors for Colleges and Universities. Chairman, Professor Van Vleck. Cf. BULLETIN, November, 1910.

No attempt will here be made to summarize these reports completely or even to refer to all questions treated in them.

† Except at Yale University, where the degree of doctor of philosophy was established in 1860.

‡ Peirce was tutor or professor of mathematics at Harvard from 1831 till his death in 1880, but, except during the period here considered, it was only in the last ten years of his life that, under the influence of an expanding elective system, he again began to have an appreciable number of advanced students.

A few of these have since attained world-wide fame, while others were influential in introducing advanced mathematical instruction into the United States twenty or thirty years later. Peirce's success in collecting at this time a fair number of competent students for graduate work seems to have been due primarily to the presence of the office of the American Nautical Almanac at Cambridge from 1849 to 1866, and, secondarily, to the founding in 1847 of the Lawrence Scientific School,* which, in those early years, possessed, under the leadership of Louis Agassiz, Jeffries Wyman, Asa Gray, and others, some of the aspects of what is now known as a graduate school.

We notice, in passing, the contrast presented at this time, and for many years after, between the increasing supply of good astronomers in this country and the lack of men who, even by a stretch of the imagination, could be called mathematicians. It may fairly be said that the mathematical talent of the country was at this time diverted to astronomy.

Various circumstances united to bring a large measure of success in the establishment of graduate instruction in all fields, and in particular in mathematics, during the years 1870–1890. The great increase of wealth in the country brought with it endowments of many sorts which strengthened the older universities and established some important new seats of learning. Three things may be mentioned which, on this basis of material prosperity, did more than anything else to help forward the cause of graduate study in the critical period we are now considering.

1. *Study Abroad.*—For many years an occasional American had gone abroad to complete his studies. Thus B. A. Gould, a pupil of Benjamin Peirce and a graduate of the class of 1844 at Harvard, who later became eminent as an astronomer, studied with Gauss in Göttingen and took his doctor's degree there in 1848. Similarly J. Willard Gibbs after taking his doctor's degree at Yale in 1863 spent three years (1866–1869) in Paris, Berlin, and Heidelberg, where he studied with Kirchhoff, Helmholtz, Weierstrass, and others. A few more cases of a similar sort might be recorded, but it was not until the end of the

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* In the same year the department of Philosophy and the Arts was organized at Yale with the purpose of furnishing "resident graduates and others with the opportunity of devoting themselves to special branches of study," these branches embracing "theology, law, medicine, and more particularly mathematical science and physical science and its applications." It was in this department that the doctor's degree was established in 1860, as noted above.

seventies or the beginning of the eighties that the stream of mathematical students from America to Europe (generally to Germany) became a steady one. This tendency to go to Germany for the closing years of study contributed probably more than anything else to build up sound standards of productive scholarship and of graduate teaching, without which all attempts to establish advanced instruction in this country must have remained abortive. Its success was in part due to the establishment and the wise administration of traveling fellowships, first at Harvard and then, to a much less degree, elsewhere. We shall return to this important matter of study abroad in a later section.

2. *The Foundation of Johns Hopkins University.*—The magnificent bequest of Johns Hopkins of three and a half million dollars for the foundation of a university in Baltimore, and his wisdom in leaving his board of trustees a free hand in the organization of the institution resulted in the adoption, on President Gilman's initiative, of a plan whereby the ordinary undergraduate instruction was relegated to a subordinate position from the very start, so that the new university stood before the American public as the standard bearer of the higher education. This was of inestimable benefit in strengthening the hands of those members of the faculties of the older universities who had been struggling to establish and develop graduate instruction at their own institutions. The presence of the eminent English mathematician, Sylvester, as professor of mathematics during the first seven years of the Johns Hopkins University, had also a marked effect in stimulating interest in advanced mathematical studies in America, though it is easy to overestimate his direct influence, as he was a poor teacher with an imperfect knowledge of mathematical literature. He possessed, however, an extraordinary personality, and had in remarkable degree the gift of imparting enthusiasm, a quality of no small value in pioneer days such as these were with us.

3. *The Elective System.*—At the beginning of the period under consideration the lack of students qualified to undertake advanced work was most keenly felt and made any large success in the establishment of graduate instruction an impossibility. The adoption under the lead of President Eliot, first at Harvard and then to a greater or less extent throughout the country, of a far-reaching elective system in the four-years' undergraduate course furnished a possibility for the gradual extension of

instruction in the special fields. Without entering on the question of the advantages and disadvantages of the elective system for the college itself, we may safely say that it provided a basis for advanced instruction without which any considerable development of such instruction, at least during the years of which we are now speaking, would hardly be conceivable.*

At the close of the period we are considering, when the idea of graduate instruction had already taken a firm hold on many of the stronger institutions of the country, the founding of Clark University exclusively for graduate study in mathematics, psychology, biology, physics, and chemistry gave a further impetus to specialization in advanced work, and the opening of the University of Chicago in 1892 may almost be said to mark an epoch in the development of graduate instruction in the West and Middle West; for, though that university had from the start an undergraduate department, it stood out, through the character of its faculty and the emphasis laid on research work, as a strong exponent of the graduate idea.

While in these universities, as well as at Johns Hopkins, advanced instruction was at once placed in a department by itself, in by far the larger number of institutions it developed very gradually within the old college, room being made for it by the elective system; and it was only slowly, even in the larger institutions, that small groups of graduate students began to collect. The somewhat unorganized condition, which was then the rule, is still to be found in the weaker institutions of the country and also in some colleges which in their chosen field of undergraduate work are strong, but which voluntarily renounce any substantial development of graduate instruction. The great universities, however, have all, since the year 1890, developed well-organized graduate schools frequented by the graduates of their own and other colleges. It may be added, to avoid possible misconception, that the graduate schools which sprang full-fledged into existence and those which developed slowly from the old college no longer form two distinct classes. Some of the strongest graduate schools in the country are now to be found among the last-named institutions.

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* Cf., however, the closing remarks of § 2. What we desire to emphasize here is that an elective system so arranged as to allow some specialization in individual departments, not merely the choice between various *elementary* subjects, permitted a gradual development of more and more advanced instruction, the students being at first mainly undergraduates. *Such* a development could go on simultaneously at many places, while even a single attempt to duplicate the Johns Hopkins experiment would probably have quickly led to disastrous failure.

In contrast with such countries as France, Italy, and, to a less extent, Germany, we note the complete lack of central control or organization in the United States. Many variations are hereby made possible which are, for a country like ours, almost a necessity; and competition, on the whole healthy, springs up between the different institutions.

In conclusion we note that of late years some technological schools (for instance the Massachusetts Institute of Technology) have undertaken a limited amount of graduate instruction in mathematics. As this instruction does not differ, except in the greater emphasis laid on applied mathematics, from that given at colleges and universities, and since the amount of such instruction at technological institutions is as yet very small in comparison to the whole amount of mathematical graduate instruction in the country, we have not thought it necessary to mention these technological schools specifically in what follows.

2. *The Graduate Student of Mathematics at the Present Day.*

Owing to the great variety of standards for the bachelor's degree in the different colleges of the country, the students of a single graduate school enter it with very diverse preparation. This is, however, not so disturbing as might be expected, owing to the fact that at every university in which a graduate school exists there is a collegiate or undergraduate department whose instruction is freely open to the graduate student who is in need of it. We may then say that not all work done by graduate students is graduate work. On the other hand, the ambitious and capable senior in colleges allowing considerable freedom of election will frequently be doing work of a distinctly graduate character in the same classes with able graduates of colleges in good standing.

If we thus miss any sharp line of demarcation at the lower limit of the graduate school between graduates and undergraduates, we find a similar phenomenon at the upper limit where the graduate student often passes by almost imperceptible steps into the teacher. Indeed there are graduate schools, even among the better institutions of the country, the bulk of whose students are at the same time assistants or instructors. This, and the very high percentage of graduate students of mathematics the country over who are fellowship and scholarship holders are features of American education which it is to be hoped, will gradually pass away.* They are closely related to the presence in graduate schools of large numbers of students of mathematics who have reached an age when their student days should be over. It can not be too strongly urged on all who give young men advice or who are influential, by awarding scholarships or otherwise, in shaping their careers that it is only in his first youth (not at the age of thirty or thirty-five) that the foundation of real success can be laid by the student of mathematics.

A somewhat different class is formed by school-teachers in active service who are at the same time enrolled as graduate students of mathematics, but at any moment take necessarily only a small amount of work. The ambition of these teachers to improve their professional equipment is most laudable. When, however, as is sometimes the case, they form a considerable proportion of the enrollment of a graduate school, they may be a source of weakness to that school in spite of their earnestness of purpose.

The period spent by a student in graduate study varies from one to three, or even more years; and the amount of migration from one university to another does not seem to be large, although the great majority of students attend a graduate school at a different university from that at which their undergraduate years were spent.

We note also that in graduate work coeducation is the almost universal rule, not only in the great state and other western institutions where coeducation forms an integral part of the scheme of education from top to bottom, but even in the most conservative institutions of the East, which do not admit women to their undergraduate departments. Apart from Princeton and the University of Virginia, where no women are admitted, it is only in women's colleges (Bryn Mawr, Vassar, etc.) and in some institutions for men which have held firmly to the undergraduate idea, so that the amount of graduate work is very limited, that one sex alone will be found.

A striking and significant fact is that nearly half of all graduate students of mathematics come from small colleges. This is probably due to the fact that in such colleges students always have the opportunity to study the elements of mathematics and often something beyond the elements, while the inducements for them to turn away into other fields are slight in comparison to those offered at larger institutions where a richer elective system prevails. The tendency so strong in our day and country to regard the man of action as being of nobler clay than the man of thought and ideas, reinforced by the much greater financial prizes open to the former, whether he be lawyer, business man, or engineer, creates a situation where it is not easy to secure for mathematical study a due proportion of the strongest youth in our college communities.

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* Nothing in any way resembling this free award of financial aid is found necessary to induce strong men to attend schools of law or engineering. Cf. the closing lines of this section.

3. *The Organization of Advanced Mathematical Instruction.**

The purpose of mathematical instruction should be fourfold:

I. To impart knowledge.

II. To develop power and individual initiative.

III. To lead the student to express adequately and clearly what he knows.

IV. To awaken the love of knowledge and to impart scholarly ideals.

The first of these aims, without attention to which the other three can not be obtained, has led to the great development of the lecture system which we find in all graduate schools, and to the use of the textbook and treatise either in connection with these lecture courses or independently of them, and of the original memoir the reading of which constitutes an art by no means easy to acquire, and which deserves special cultivation at the hands of the members of the teaching staff.

As means used under II may be mentioned: first, the solution of problems by students either in connection with the lecture courses or in special seminars or proseminars, and secondly, the writing of theses which may or may not be connected with the doctor's degree. This last is also the chief means employed under III, although the quiz (cf. Subcommittee 3, § 4) is sometimes employed effectively for this purpose, and even the brief written problem is not without some value here.

Both II and III above should receive more attention than is now commonly given to them, while I is at present adequately treated, except, perhaps in the relative weakness of applied as distinguished from pure mathematics.

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* Cf. also the report of Subcommittee 1.

The aim indicated under IV depends for its attainment less on special methods of instruction and more on the personality of the instructor and his attitude towards science than do I, II, or III. To secure adequately the end in view, an instructor is needed who combines high *scientific* ideals with a commanding or sympathetic personality. Such men could do much to counteract the tendency noted in the closing lines of § 2, but, on the other hand, it is precisely this tendency which makes them difficult to secure. Their influence on undergraduate instruction should be no less valuable than in the graduate school.

We must come back once more to the lecture courses which everywhere form the backbone of graduate mathematical instruction. Such a course usually extends either through the whole academic year, that is from the end of September till early in June, or through the first or second half of this year.* The lectures, of somewhat less than an hour each, come usually three times (less frequently twice) a week. While much depends on the local traditions and the personality of the instructor, it may in a general way be said that these lectures have a far less formal character than is the case in European universities. Students will frequently interrupt the lecturer with a question, and short discussions between the instructor and one or more students will not infrequently take place, and at times the more formal quiz (cf. Subcommittee 1, § 6 and Subcommittee 3, § 4) finds its place here. Some tact and firmness are occasionally necessary to prevent the loquacious or thick-headed student from monopolizing the time of the class, but on the whole this frequent contact during the lecture between teacher and student is an admirable feature of American higher education, and counteracts, to a certain extent, some evils which usually accompany the lecture system. It is made possible by the smallness of the classes, an audience of twenty-five in a graduate course in mathematics being distinctly unusual.

The range of subjects covered by the courses offered in each graduate school is very great (cf. Subcommittee 1, § 5). This is peculiarly the case in those institutions which have only recently begun a policy of expansion in their graduate work, where the first sign of such expansion often appears in an astounding increase in the number and range of courses offered, for only a small part of which there are students. Indeed, if students should present themselves, the capacity of the teaching force would be completely overtaxed. This is a state of affairs which no self-respecting institution should allow to continue, and there are signs that it is usually of only a temporary nature, since with a real strengthening of the mathematical department of such an institution this inflation tends to disappear. We hasten to add that the stronger institutions, and many smaller institutions with a due sense of proportion, offer admirable selections of courses commensurate with their capacity and the needs of their students, courses which at each institution usually vary considerably from year to year. Even in the weaker institutions where a call for advanced instruction is hardly apparent, it may often be wise to encourage instructors to offer a course of a not wholly elementary character, as it will frequently be found to act as a tonic and, by keeping them in touch with the scientific side of their subject, enable them to make their elementary work more vital.

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* Attention must also be called to work in the summer schools and summer quarters. Cf. Subcommittee 1, § 4.

It was mentioned in § 2 that no sharp distinction between graduate and undergraduate work in mathematics can be made. Indeed it is hard to exclude entirely from graduate work anything above the first course in the calculus, now commonly taken in the second undergraduate year. The actual state of affairs is best expressed by regarding the group of courses just following this point, such as a second course in the calculus, the elements of determinants and of the theory of equations, projective geometry, a first course on differential equations, etc., as belonging both to graduate and to undergraduate instruction. From this latter point of view, however, these courses usually appeal only to the student of distinct mathematical ability and seriousness of purpose, whose presence in the course along with graduates does not very greatly affect the character of the course.

As the external signs of success for the graduate student we have the master's and the doctor's degrees. The first of these is commonly given for one year's graduate work done largely in one subject, such as mathematics or physics, and tested either by course examinations in which a higher standard is demanded than is accepted for undergraduates, or by a single examination covering the whole year's work. A thesis is also often required for the master's degree; but the work done on this thesis is not commonly of the nature of research work, and the degree is taken by considerable numbers of students most of whom never proceed further. This degree is given, and properly given, by a large number of institutions, many of which have only a very moderate strength in their graduate mathematical work. Under these conditions suggestions for a minimum standard for the degree are not out of place, and such suggestions will be found in the report of Subcommittee 1, § 7.

The doctor's degree originally came to us from Germany, but has long been naturalized and is in all American institutions of good standing distinctly a research degree. In several of our stronger universities it has a standard at least as high as the best German standard. The requirements for the doctor's degree in universities which have been giving it to any extent during the last ten years are tolerably uniform (cf. the report of Subcommittee 2), but in this matter so much depends on the unwritten standards of individual professors or departments that there still remains a great difference in the ease with which the degree can be obtained at different institutions. It is for this reason that the suggestion which is sometimes made that it would be well to attempt to formulate definite standards for the doctor's degree to which the universities of the country should conform, seems to be of slight practical value.

In school and college work America adopts in one respect a very different standpoint from France and Germany, and this has a certain indirect influence on graduate work. We refer here to the fact that in the last-named countries a pupil will not be allowed to proceed from class to class, and, particularly, will not be allowed to pass the great educational landmarks (for instance graduation from the gymnasium in Germany) without conforming to a very exacting standard which a considerable percentage of each class fails to attain. In America, on the other hand, the teacher who tries to impede seriously the progress of any but the unusually lazy or stupid soon makes himself impossible. This is not the place to discuss the respective merits of these two points of view in the secondary school or even in the college; but when we come to the graduate student of mathematics it seems clear that the American attitude must be modified, and, as a matter of fact, in all the stronger institutions of the country a much greater ability and earnestness of purpose is demanded for passing examinations and securing degrees in the graduate school than would be allowed to pass muster in undergraduate work. Nevertheless, it is to be hoped that something more will be accomplished in this direction, and that, in particular, candidates for the doctor's degree will be made to feel that success for them at an institution of good standing is not a mere matter of time and patience. It is the more important to insist on this since, as has just been said, the whole current of secondary and college education runs in another channel.

4. *Teachers.*

We must be concerned with this subject for two different reasons: first, because the great majority of graduate students of mathematics ultimately become teachers in secondary schools, colleges, or universities; and, secondly, because on the quality and efficiency of the teachers in the graduate school itself (professors, instructors, etc.) depends to such a large degree the quality of the school.

It is a favorable sign of the gradual elevation of the profession of secondary-school teacher that of late years many persons wishing to adopt this profession spend a year in study in a graduate school. It is true that this time is frequently not spent in the study of a single subject; but for the future teacher of mathematics (or of mathematics and some other subject) to have had a couple of graduate courses in mathematics, usually in the intermediate group referred to on p. 130, is a very substantial gain over the conditions of twenty years ago. It is to this class of students that the courses on the teaching of mathematics, which are now given at many colleges and universities, mainly appeal.

If we except this group who go into secondary-school teaching, and a second group who study mathematics as a tool for use in some other science, such as physics, it may be said with almost absolute precision that all other students of mathematics in graduate schools become instructors in mathematics in colleges or universities. The condition of twenty-five years ago, where college instructors in mathematics were taken from among the freshly graduated students of a college (usually the same college where they were to teach), has now become the exception instead of the rule; and where it still occurs, the appointment is usually a temporary one, both the instructor and the college expecting that, after a year or two of teaching, further graduate study will follow. The gain involved in this changed state of affairs, both in breadth of view and in real mastery of the subject, the teaching of which is to be the young man's life work, is so obvious as to require no further comment here. If the student can furthermore be given some comprehension of the fact that the science of mathematics is a living and growing one through contact with other students or instructors who are themselves contributing to this growth, and still more if he himself can take some part in the development of mathematical knowledge, his outlook on mathematics in particular and intellectual life in general will have been so broadened that he can hardly fail to become a better member of a college faculty than would otherwise have been the case.

After all this has been said, we must, however, admit that this question has also another side less pleasant to contemplate. What passes for original research, in this country more even than abroad, is often hardly a real contribution to mathematical progress at all, but merely a grinding out of results, which if they have only never been published before may be as unimportant and unattractive as you please; they form an "original contribution." One is tempted to answer: Yes, in the same sense as the brass button in the contribution box. We may feel certain that in the long run this will be the character of the research work done by students who have no real capacity or inclination for original work, but who are pushed into it by the increasing demand, on the part of certain heads of departments, for the doctor's degree as a necessary preliminary to college teaching. The pressure thus produced will surely, if persisted in, bring forth an increasing yearly crop of doctors,—success can be obtained by almost anyone with a fair mathematical capacity and with sufficient industry and patience, either by going abroad or by going to one of the weaker American institutions with an ambition for giving the doctor's degree. It is doubtful if the time will ever come, certainly it will not come for a great many years, when all the members of the teaching staffs of the large universities of the country, and the colleges of like rank, can be men with a real capacity for original investigation; the number of all such men in the country falls far short of (one might almost say that it is of a different order of magnitude from) the number of places to be filled.

The pseudo doctor, to whom reference was made above, is often narrowed rather than broadened by the bit of investigation which he has been set to do, and becomes thereby less effective as a teacher, investigation for him becoming a fetich for which he forgets all other ideals. Or, on the other hand, he may let all thought of original work drop out of his mind when once he has secured his degree. In either case the letters he places after his name ought not to go very far in recommending him for teaching positions. A broad and deep mathematical training should surely be demanded by all the institutions of the country which claim collegiate rank as a prerequisite for a permanent appointment on their teaching staff. They will naturally demand also some ability as a teacher. If in addition they can secure an investigator of a genuine sort, even though his caliber be slight, they should usually regard themselves as fortunate, though a few of the strongest institutions can and should set themselves a much higher standard. On the other hand, our stronger graduate schools should continue, as they are now doing, to encourage every capable student to try his hand at some piece of original investigation, but they should not hesitate, after a fair trial, to tell him, if that turns out to be the case, that he is not fitted for that kind of work.

No specific training for the profession of college or university instructor is commonly given in graduate schools apart from the training in mathematics (cf. Subcommittee 3, § 5). The statement made in § 3 of the present report that the training in clear and adequate exposition which is given to graduate students of mathematics is frequently insufficient is of peculiar importance in relation to the future teacher. While it is probably not desirable to attempt to train the future college or university instructor in the art of teaching, the question whether more can not be done to lead graduate students of mathematics to express their ideas well both in spoken and in written form is worthy of serious consideration.

Let us turn now from the graduate students, who are to become college instructors, to the actual instructors and professors of mathematics in our colleges and universities. If we compare conditions at the present day with those existing twenty years ago, a very great increase in the standard of mathematical knowledge on the part of the teaching staff is evident. That the improvement here has not been even greater is due in large measure to the fact that the supply of well-trained graduate students falls far short of the demand. Weak appointments are also made from time to time, owing to ignorance on the part of trustees or heads of departments of what really constitutes a mathematician, to the pernicious view that administrative ability may be allowed to take the place of mathematical ability, or to other like causes. Flagrant cases of this kind occasionally occur which make one blush for the good name of American universities, but such cases are now merely sporadic and one gains comfort by contemplating conditions in Germany only a hundred years ago. What is needed here, as in so many other places in American life, is a strengthening of *intelligent* idealism (we have more than enough misdirected idealism amongst us) based upon knowledge, and there seems every reason to hope that the great development of mathematics in this country during the last twenty years, evident chiefly in the growth and activity of the American Mathematical Society, will in an ever-increasing degree supply the intelligent and influential public opinion here needed. The shortage, above mentioned, in the supply of instructors in mathematics forms the most serious aspect of the situation.

For various further points: The excessive burdening of young instructors with drudgery, which still often occurs; inadequate salaries; the burdening of professors with administrative work; we refer to the report of Subcommittee 3.

5. *Study by Americans Abroad.*

No account of higher mathematical education in America would be complete without a reference to the part played by the study of Americans abroad. What an important factor this was in introducing advanced mathematical instruction and research into America has already been mentioned in § 1. In the early days the possibilities for advanced mathematical study in this country were very limited, so that it was natural that students able to do so should go abroad where they could find this opportunity in large measure. At the present day it may safely be said that at several of the stronger American graduate schools most American students find mathematical opportunities better suited to their needs than are to be found at any place abroad. Nevertheless, students still go abroad in apparently undiminished numbers to study mathematics,* and their decision to do this is frequently a wise one. Let us inquire how this can be the case.

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* It would be a matter of considerable interest to have statistics on the number of American students who go abroad each year to study mathematics and the length of time they stay. Such statistics would seem to be very difficult if not impossible to secure. As to the proportion of instructors of graduate courses in mathematics who have spent at least one year abroad, see the report of Subcommittee 1, § 3. Far less important is the question of the number of doctor's degrees conferred on Americans

There come first considerations of a non-mathematical character. It is desirable for everyone to become acquainted at first hand with other countries than his own, and this is doubly true for an American, for whom a period of residence in European countries is invaluable. It is true that the student often seems to have brought back from a year or two of residence abroad only a strengthening of his earlier national prejudices, since the mote in the neighbor's eye is so very easy to discern;* but if he is worth his salt, he brings with him a fund of impressions and experiences which, as time goes on, greatly enrich his life. For this reason alone study abroad is to be recommended even at some mathematical sacrifice. A second consideration is that the cost of living in Germany, to which country the great majority of students going abroad have always resorted, even after the great increase of the last few years, is still lower than in America, and in particular, the tuition fees are much less than in many of the larger American institutions, especially of the East. These facts largely counterbalance the expense of the trip across the ocean. Finally, it is to be remembered that a year or two of mathematical study in Germany, France, or Italy gives the student a reading and speaking knowledge of one of the great languages of modern thought, besides his own native English, such as can hardly be acquired in any other way.

When we come to mathematical considerations, the first question we must ask is whether getting a degree or learning mathematics is the prime object of the student going abroad. It is the former which, owing to circumstances mentioned in § 4, is too often uppermost in his mind. A student of this category had much better go abroad for his degree than to a second-rate American institution. Of course some care must be exercised

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abroad. Such information might be secured. We content ourselves with giving two such items, for which we are indebted to Dr. Dunham Jackson:

At Göttingen in the years 1889–1909, inclusive, 22 Americans received the degree in mathematics, while no degrees in mathematics had been conferred on Americans during the four previous years.

At Leipzig in the years 1885–1902, inclusive, 8 Americans received the degree in mathematics, while after this time Americans seem to have ceased taking the degree in mathematics there.

At present from 2 to 4 Americans take their degree in mathematics in Germany each year, as against an average of 16 or 17 in the United States.

* There are also cases in which he takes so kindly to foreign conditions as to become out of touch with America. It is, however, rare that this state of affairs should survive his return more than a few months.

by him in the choice of his university, or he must have good fortune in writing a thesis whose weak points are not evident on a superficial examination, but his task is, on the whole, not a difficult one, and he gets at least the advantage of a period of foreign residence.

For another class of men foreign study may be recommended without qualification, namely, for able students who have already had a substantial training in one of the better American graduate schools, or who have even taken the doctor's degree at such a school. Such men will naturally go either to one of the great mathematical centers like Paris or Göttingen, where they will have the opportunity to hear lectures by several of the leading mathematicians of the day, and, perhaps, to see some of them occasionally outside of the lecture room; or they will select some mathematician of eminence in a particular field with whom they may hope to gain direct personal contact, and go to the university where he happens to be. Thus of late years a small but steady stream of American students has gone to Italy.

To the students just considered, and to some extent to their weaker comrades mentioned above, the period of residence at a great European mathematical center or of contact with an eminent mathematician at a less important European institution brings with it a realization of what high scientific ideals in mathematics are, and to what an extent they prevail abroad. Such ideals prevail also, it is true, at the strongest American institutions; but it is hard for the young American to appreciate their great diffusion in a ripened civilization until he has experienced it by personal contact.

SHORTER NOTICES.

Shop Mathematics. By EDWARD E. HOLTON, Head of the Department of Machine-Shop Practice, the Technical High School, Springfield, Mass. Springfield, The Taylor-Holden Company, 1910. xi+211 pp.

THE present rapid development of secondary vocational schools and their competition with general secondary schools have set in operation forces which tend to modify materially the character of secondary mathematical teaching in this country.

Since the mathematical work of the higher institutions rests upon the foundation laid by the secondary schools, it is important that any tendency to alter that foundation should be fully appreciated.

The book under review is one of the first published results of the attempt to develop a course in mathematics adapted to the needs of the secondary technical school, and as an indication of those needs, and possibly of the character of the mathematical instruction in such schools, it deserves serious attention.

The pioneers of the movement of protest against the traditional curriculum and methods of the secondary school have set high standards for the work of their own institutions. Witness the statement of Dr. C. M. Woodward:* "No blow is struck by him" (the pupil), "no line drawn, no motion regulated by mechanical habit. The only habit acquired is that of thinking. The quality of his every act springs from the conscious will, accompanied by a definite act of judgment."

Let us test the book before us by this standard, remembering that its object is to teach the pupil "what the shop problems are"† and "how to apply mathematical principles, rules, and formulas to the solution of such problems" and that "no attempt is made to teach mathematical theory or principles."

Consider the following problem (Problem 17, page 28). "What length of bar will be required to raise a building of 100 tons weight with 10 lbs. pull each on 100, ¼ in. lead jack-screws?" Result (page 191), 7.957 inches. If the reader, or better, the author, will try a simple experiment with a spring balance he will find that a force of ten pounds can easily be exerted by merely bending the little finger. The length found for the bar is about an inch more than that of an ordinary penholder. Now just picture a hundred laborers at work raising a building, each pulling with the little finger of one hand on a bar an inch longer than a penholder. Then picture what happens to the "shop foreman of twelve years' experience" when the contractor who is paying the gang of one hundred laborers, comes around and sees what they are doing! Again, consider the computed result: seven and nine hundred fifty-seven thousandths inches. Remember this is a problem on the jack-screw, and the power is applied, not by means of a knife edge, but by the hand. The author gives the length of the bar to the thousandth of an inch!

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* Report of the U. S. Commissioner of Education, 1893–4, part I, p. 896.

† Introduction, p. viii. This introduction is by Dr. C. F. Warnier, principal of the Springfield, Mass., Technical High School and of the Evening School of Trades.

This is not an isolated but a typical example of the "definite act of judgment" which accompanies the author's consideration of the data and results of his problems. For example, in problem 12, page 51, the pupil is asked to find the weight which can be raised by a force of 60 lbs. applied to a somewhat complex combination of screw and gears. The answer is given (page 195) as 203,575.68 lbs., i. e., to one part in twenty million. If the given force of 60 lbs. be altered by an amount equal to the weight of a single drop of water the computed result will be affected by more than the 0.68 lbs. appearing in the answer. This sort of thing is not a result of mere inadvertence in writing down the answers; it appears in the illustrative examples which are intended to serve the pupil as models. For example, in a problem concerning the friction of a lathe (page 54), where the coefficient of friction is given as .08, we have the following work:

$$400 \times .08 = 32, \quad \frac{32 \times 1\tfrac{1}{2}}{12} = 4,$$

$$\frac{50 \times 2\pi r}{12} = 50 \times 6.2832 = 314.16,$$

$$\frac{314.16 \times 4}{33000} = \frac{1256.64}{33000} = .038.$$

The coefficient of friction is known here only to one part in 8 but π is used to five significant figures, i. e., to one part in thirty thousand.

The pupil accustomed to books "written almost entirely from the point of view of the teacher of pure mathematics with little reference to concrete problems of life and having no reference whatever to the actual problems of the drafting room and the shop"* may be pardoned if he asks for information concerning the concreteness and actuality of problems, results, and methods of which the above are samples. With such models before him is the pupil likely to acquire the habit of thinking?

The author seems not to have realized that the standards of efficiency governing the mathematical work of the shop are entirely analogous to those by which the mechanical work is judged.

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* Cf. Introduction, p. vi.

In a little Handbook for Apprenticed Machinists issued by a prominent firm of manufacturers primarily for the use of their own apprentices the following statement appears: "He" (the apprentice) "should know the parts that are to be accurate; if a part is machined only to make it smooth, there is no need to take time to size it accurately." Had the author caught the spirit of this and applied it to the construction of his book, he would not have given the length of a jack-screw bar to the thousandth of an inch or a weight of a hundred tons to the hundredth of a pound.

The author's mathematical methods exhibit the same lack of appreciation of the essential features of a problem as is displayed in his data and results. For example, a problem in spiral gearing (page 45) involves the following relations:

$$P_1 = \frac{\pi}{P}, \quad P = \frac{P_1}{\cos Y}, \quad N = \frac{\pi D}{P}.$$

The data are $P = 16$, $D = 1\frac{1}{2}$, $Y = 60°$; the required quantity, N. The author's solution is

$$P_1 = \frac{3.1416}{16} = .19635,$$

$$P = \frac{.19635}{\cos 60°} = \frac{.19635}{.5} = .3937.$$

Then

$$\text{No. of teeth (i. e., } N) = \frac{1\frac{1}{2} \times \pi}{.3937} = \frac{4.7124}{.3937} = 12.$$

Could a better example be found of how *not* "to apply mathematical . . . formulæ in the solution of such problems"?* In the three pages of explanation of the principles of spiral gearing which the author has prefixed to this solution it has not occurred to him to collect the formulas bearing upon it and to deduce the simple relation $N = DP \cos Y$, which gives at once, on inserting the data, $N = \frac{3}{2} \cdot 16 \cdot \frac{1}{2} = 12$.

This particular performance is especially provoking in that it violates the cardinal principles both of efficient and of accurate shop practice. Efficiency calls for the avoidance of unnecessary operations, accuracy, the restriction as far as possible of the number of operations of adjustment, e. g., the centering of a piece of work in the lathe. In the example quoted the author has violated the first principle by executing processes which are equivalent to multiplication followed by division by π; he has violated the second in that both these operations are necessarily effected with approximate numerical values and are consequently unnecessary operations of adjustment.

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*Introduction, p. viii.

The mathematical content of the book consists largely in the statement of formulas and the substitution of numerical data therein. As indicated in the problem on spiral gearing, the author does not make efficient use of even the small amount of elementary mathematical knowledge which may reasonably be supposed to be in the pupil's possession. The notation used is frequently made unnecessarily cumbersome by the use of two or more letters where one would be sufficient, e. g., *Wa* for weight arm (pages 8, 10), *CP* for circular pitch (page 33). This last notation seems a wholly unnecessary and undesirable complication. Not only do both *C* and *P* appear in the problem, but their product is not the circular pitch which the author calls *CP*. Moreover the notation used in practice, in so far as the reviewer can judge from the publications of the manufacturers, and from a widely used Mechanical Engineers' Pocket-Book, to both of which the author acknowledges obligation, is not that adopted by him.

The introduction contains the following statement* "What is needed is to purge the old books of useless material and put in place of it practical mathematical work distinctly planned to make up for the shortcomings of the old methods when measured by the practical demands of modern times." It would be impossible to state more accurately the process to which this book should be subjected.

The vocational schools have already attained a prominent position in our system of education. For the interest of these schools themselves as well as for that of the other secondary and the higher schools it is important that their mathematical instruction shall be of a high standard. As to the precise content of the curriculum there is room for much honest difference of opinion. When however we consider that the problems of the shop demand besides the elementary operations such things as geometric progression (cone pulleys and hoisting tackle), cube root (belting problems), Euclid's algorithm of the greatest common divisor and continued fractions* (gearing and screw problems), it would appear that if the course in mathematics is to do nothing more than provide a secure foundation for the work in the shop its abstract content is not likely to be much less extensive than has been usual in secondary schools. The nature of the concrete problems will naturally vary with the requirements of the different schools. But whatever their nature they must be *real* shop problems, that is they must be such that in solving them the pupil is compelled to consider not only the purely mathematical element but also the significance and reasonableness of the numerical data and results and the appropriateness of the algebraic and arithmetic processes used in their solution. Excessive formalism has been the bane of the teaching of abstract mathematics. It is just as common and just as pernicious in the shop as in the class-room.

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* Introduction, p. viii.

CHARLES N. HASKINS.

Die Mathematik in den physikalischen Lehrbüchern. Von H. E. TIMERDING. Band III., Heft 2. Leipzig, Teubner, 1910. vi + 112 pp.

IN the systematic study of the teaching of mathematics in Germany which is being made under the auspices of the International commission of the teaching of mathematics, the present volume covers the field of the mathematics required and used in the physics of the "Höhere Lehranstalten" and "Hochschulen." The principles of mathematics found in the text-books on physics in use in Germany to-day form the basis of the discussion.

While the author states that the mathematics of physics is mainly of a geometric nature, it is easily seen that he considers the main problem to be concerned with the amount and quality of the infinitesimal analysis used in the texts investigated. Fundamentally, the problem may be stated as follows: The exact theory of most physical phenomena in its development requires a use of the principles of infinitesimal analysis. A scientific attitude toward these problems on the part of the instructor will not allow him to be satisfied with mere formulas or even a confused word picture, or the skillful manipulation of a "near calculus" which may interest but not convince the student. The instructor in physics may not count on a thorough training on the part of all his students in the rigid proofs of these fundamental principles. What ought to be his attitude concerning the use of these principles?

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* Practical Treatise on Gearing. Brown & Sharpe Mfg. Co., pp. 130–134.

As illustrations we present the cases of falling bodies and moment of inertia. They are both included in a course in physics. The former requires either two differentiations or two integrations; while the exact theory of the latter requires fundamentally the principles of integration. The former is usually handled by the aid of the notions of average velocities; while, in the case of the latter, in the expression $\sum mr^2$ the size of the m is — in some cases — carried down to the molecule. Most texts do not go to the limit. Hence the author's statement that the important thing is not what is said but what is left unsaid and that the instructor, if honest, must clearly recognize the necessity of the principles of the infinitesimal analysis. The statement is further made that one reason why so many subterfuges, instead of the symbols of the calculus, are used is because the authors wish their books to sell. The notions of the infinitesimal analysis would not confuse the subject but clear it up. Calculus symbols should not be used as loosely as they are. It is an evidence not of good pedagogy but rather of downright deceit not to use them.

An excellent chapter on the history of physics and its problems, from the beginning of the seventeenth century when it was a branch of the Aristotelian philosophy, to the present with its wondrous development, is included. The correlation of mathematics and physics is discussed, the influence of physics in enriching the mathematics and of the latter in putting physics on a safe foundation of rigorous proofs.

It is pointed out that Germany has lagged behind both England and the United States in the use of such aids as graphs, accurate drawings and illustrations, and others suggested by technology, which help to give the student clear and definite notions of mathematical or physical problems.

The problem of the requisite instructions in mathematics presented from the viewpoint of the teacher of physics is clearly stated and a scientific attitude is called for. What the nature of the correlation of instruction should be the author but slightly suggests. It would seem to call for a training at an earlier period in the principles of the infinitesimal analysis. This correlation is being worked out today.

Ernest W. Ponzer.

Practical Measurements. By A. W. SIDDONS and A. VASSALL. Cambridge, The University Press, 1910. xiv + 60 pp.

THE definite purpose of this little note-book is to furnish for the boys at Harrow a series of elementary experiments in measuring and weighing, conforming with the recommendations embodied in the recent report on "The correlation of mathematical and science teaching" drawn up by a joint committee of the mathematics and science teachers of England.

On the mathematical side the experiments deal with the measurements, correct to three significant figures, and the necessary computations — of lengths, areas, and volumes — a practical checking up on the theorems of geometry.

The experiments emphasizing the side of physics deal with the weighing of liquids and solids and the determination of their specific gravities. Experiments with the hydrometer, barometer, pendulum, the U tube, and on Boyle's law are included. In the appendix experiments with the vernier, micrometer screw-gauge, and spherometer are added.

The methods used throughout the book are its strongest feature. Boys thoroughly grounded in the fundamental principles emphasized throughout will be able to do their future work with greater efficiency. Accuracy in estimating, measuring, computing, laying out to scale, and orderly arrangement are required. Concrete results, which should check up with theory, are required throughout. The practice of requiring such definite results, correct to a certain number of significant figures, is well worth while.

ERNEST W. PONZER.

Azimuth. By G. L. HOSMER. New York, John Wiley and Sons, 1909. v + 73 pp.

IN ordinary engineering practice the determination of the azimuth of some line of a survey by astronomical means is not necessary; and in the cases where it is necessary, or at least advisable, the determination is likely to be a serious matter for many a practical surveyor—not because of any inherent difficulty in making or reducing the observations with all the accuracy that is needed in the field, but because the available methods are scattered in books which stay at home. Hence the utility of Hosmer's Azimuth, wherein are collected and explained with all simplicity various practicable field methods of fixing the true bearing of a line to minutes (or even to within a few seconds). In so short space may ample descriptive matter and illustrated examples be put that only about half the book, small as it is, is used for the purpose; the other half consists of the requisite astronomical, physical, and mathematical tables. The surveyor who is no astronomer need have no fear. The author requires of him no knowledge of theoretical or practical astronomy or of uranography—except that the totally ignorant might have difficulty in picking out Markab (α Pegasi) from the chart on which there are two α's in the great square of Pegasus.

E. B. WILSON.

EISENHART'S DIFFERENTIAL GEOMETRY.

IN reviewing Professor L. P. Eisenhart's book on Differential Geometry in the June number of the BULLETIN, I criticized four equations which occur at the bottom of page 313 without noticing that the arc is supposed to be the parameter along the locus of the centers of the spheres, as is stated in the last paragraph of page 310. The equations are correct as they stand with the exception of a plus sign between the parentheses in the third, which should be minus.

Professor Eisenhart has sent me the following list of errata, the publication of which may be of convenience to readers of the book:

P. 51, Ex. 24 is incorrect.

P. 87, omit the last line.

P. 117, to the expression for $\frac{d^2x}{ds^2}$ add $\frac{\partial x}{\partial u}\frac{d^2u}{ds^2} + \frac{\partial x}{\partial v}\frac{d^2v}{ds^2}$.

P. 123, last paragraph is incorrect. The position of the surface relative to the tangent plane depends upon the character of the terms of third and higher orders.

P. 179, in (86) the signs + should be −, and in the next equation the signs − should be +.

P. 180, line 27, change S to S_2.

P. 221, line 24, the upper limit of the second integral should be u_2 instead of u.

P. 241, in (52) change y_1 to y_0'.

P. 259, line 3, in place of "congruent" read "superposable by a translation."

P. 268, Ex. 7, change the last clause to "determine the two lines of striction."

P. 278, line 8, insert after "equations" the phrase "and §§ 77, 82."

P. 280, line 10, after "values" insert "(cf. §§ 77, 82)."

P. 313, in the expression for ζ change + before $(u^2 - \frac{1}{2})$ to −.

P. 400, line 9, change ω_1, ω_2 to $\omega_1 + \frac{\pi}{2}, \omega_2 + \frac{\pi}{2}$.

P. 412, Ex. 6, change + before ds_1 to −.

P. 418, line 19, remove the sign Σ after m^2.

P. 423, Ex. 9, in the equation change λ to λ^2.

P. 441, line 27, after "zero" insert "in fact L vanishes identically."

P. 444, Ex. 15, in last term of the equation change + to − and p to p_1.

G. A. BLISS.

NOTE ON COLLINEATION GROUPS.

SINCE the publication of my paper on collineation groups in the *Transactions* (volume 12, No. 2, April, 1911) my attention has been called to a similarity which exists between my determination of the collineation groups in the ordinary plane and that given by Valentiner ("De endelige transformations-gruppers theori," *Videnskabsselskabets Skrifter*, 6 Raekka, Copenhagen, 1889).

The general outlines of the first parts of the two papers are the same, as in both the groups which contain homologies of higher period than 2 are first discussed. A determination of those groups leaving a line invariant which must contain an homology of period 2 having that line for axis is given in both papers. The proofs that no group can contain homologies of higher period than 5 or homologies of period 5 are essentially the same. The proofs that no group can contain homologies of period 4 are somewhat different. Valentiner's discussion of groups containing homologies of period 3 is inaccurate and as a consequence he overlooks the existence of the G_{216}, although he considers the possibility of a group of this order.

The rest of the discussion is in the main quite different. I proved a theorem concerning the order of abelian groups contained by non-abelian groups of three times their order which is similar to his, but made no use of it in the determination of the groups in the ordinary plane. He arrived at the G_{168} and the G_{360} as the result of the solution of a diophantine equation (the same, in fact, as was used by Jordan in his attack on the problem, *Crelle,* 1878), whereas I arrived at them from the consideration of groups which left conics invariant. I however made use of a special type of this equation in two or three places.

H. H. MITCHELL.

NOTES.

THE annual meeting of the AMERICAN MATHEMATICAL SOCIETY will be held in New York on December 27–28. The winter meeting of the Chicago Section will be held at the University of Chicago on December 29–30. Titles and abstracts of papers to be presented at these meetings should be in the hands of the respective secretaries by December 9. Abstracts intended to be printed in advance of the meeting should be sent in as early as possible.

THE Annual Register of the Society is now in preparation and will be issued in January. Blanks for furnishing necessary information have been sent to the members. Early notice of any changes since the issue of the last Register will greatly facilitate the work of the Secretary. The Register is widely circulated and it is desirable that the information which it contains should be accurate and reliable.

THE concluding (October) number of volume 12 of the *Transactions of the American Mathematical Society* contains the following papers: "On the limit of the degree of primitive groups," by W. A. MANNING; "Isomorphisms of a group whose order is a power of a prime," by G. A. MILLER; "On minimal lines and congruences in four-dimensional space," by J. EIESLAND; "Volterra's integral equation of the second kind, with discontinuous kernel. Second paper," by G. C. EVANS; "One-parameter families and nets of plane curves," by E. J. WILCZYNSKI; Notes and errata, volumes 10 and 11.

THE opening (September) number of volume 13 of the *Annals of Mathematics* contains: "A method of solving linear differential equations." Second paper, by P. A. LAMBERT; "Duality in projective geometry," by N. J. LENNES; "A fundamental parametric representation of space curves," by L. P. EISENHART; "Generalization in the theory of numbers and theory of linear groups," by MILDRED SANDERSON; "Transformations of series by means of functions admitting a recurrent relation," by W. C. BRENKE; "A theorem on (m, n) correspondences," by L. I. NEIKIRK. With the present volume, the *Annals* appears in a new dress, with narrower margins and a change in style of type. The editors are Ormond Stone, Maxime Bôcher, G. D. Birkhoff, L. P. Eisenhart, Elijah Swift, Oswald Veblen, and J. H. Maclagan-Wedderburn.

THE first number has appeared of the *Tôhoku Mathematical Journal*, edited by Professor T. HAYASHI of the Imperial University, Sendai, Japan, with the cooperation of M. Fujiwara, F. Ishiwara, T. Kubota, and K. Ogura.

THE Italian society for the advancement of science held its annual session in Rome during the week October 12–18. As this is the fiftieth year of Italian independence, it seemed appropriate to give a summary of the progress made in science by Italians during the last fifty years. The following reports were made in pure and applied mathematics: "Astronomy," by E. BIANCHI; "Geodesy," by V. REINA; "Theory of functions and integral equations," by G. LAURICELLA; "Geometry," by U. AMALDI; "Mathematical physics," by T. LEVI-CIVITA; "Electricity," by M. O. CORBINO. The following general lectures were also delivered: "Evolution of the measurement of space and time," by G. CASTELNUOVO; "The new physics," by A. RIGHI; "Science in the far east," by G. VACCA; "On algebra of logic and some transcendants," by G. ANDREOLI; "Historical and critical note on the principles of mechanics," by R. GIACOMELLI.

THE sum of 20,000 rupees (nearly $7000.00) was recently appropriated by the University of Calcutta for the purpose of publishing certain ancient Indian mathematical manuscripts with an English translation.

GINN and Company announce the early appearance of a work on Advanced Calculus, by Professor E. B. WILSON, of the Massachusetts Institute of Technology.

THE reports of the American committees and subcommittees of the International commission on the teaching of mathematics are being published by the U. S. Bureau of Education, from which copies may be obtained gratis on application. The following reports have already appeared: Graduate work in mathematics in universities and in other institutions of like grade in the United States. Undergraduate work in mathematics in colleges of liberal arts and universities. Examinations in mathematics other than those set by the teacher for his own classes. Mathematics in the technological schools of collegiate grade in the United States. Training of teachers of elementary and secondary mathematics.

UNIVERSITY OF PARIS. The following mathematical courses are announced for the semester beginning November 3, 1911:—By Professor G. DARBOUX: Differential geometry, quadratic differential forms, applications of the calculus of variations, two hours.—By Professor E. GOURSAT: Differential and integral calculus, elements of the theory of analytic functions, two hours.—By Professor E. BOREL: Elliptic functions, one hour.—By Professor P. APPELL and M. CL. GUICHARD: General laws of equilibrium and motion, two hours.—By MM. E. CARTAN and MONTEL: General mathematics, first part, two hours.—By Professor H. POINCARÉ: Theory of newtonian potential and its applications to the figure of planets, two hours.—By Professor J. BOUSSINESQ: Mathematical theory of elasticity of solid bodies, two hours.—By M. A. CAHEN: Indeterminate analysis of the second degree with two unknowns, quadratic forms and algebraic numbers, two hours.

Conferences will be conducted by MM. LEBESGUE, GUICHARD, CARTAN, MONTEL, SERVANT, and ROUBAUDI.

In the Ecole Normale. By Professor E. VESSIOT: Differential and integral calculus.—By Professor E. BOREL: Theory of functions.

In the second semester, beginning March 1, the following courses are announced:—By Professor E. PICARD: Fourier's series and its generalizations.—By Professor E. GOURSAT: Ordinary and partial differential equations.—By Professor P. APPELL: General laws of motion of systems, analytic mechanics, hydrostatics and hydrodynamics.—By M. E. CARTAN: Analysis and mechanics.—By ———: Indeterminate analysis of the second degree.

In the Ecole Normale the courses of Professors Vessiot and Borel are continued from the first semester and there is added: By Professor J. HADAMARD: Mathematics.

THE following courses in mathematics are announced at the German technical schools (and at Delft, Holland) during the present winter semester. The course designated as "higher mathematics" refers to a combination of algebra, analytic geometry, differential and integral calculus, continuing for from two to four semesters I, II, etc.

AACHEN.—By Professor O. BLUMENTHAL: Higher mathematics, II, six hours; Theory of aviation, four hours.—By Professor E. KÖTTER: Descriptive geometry, six hours; Graphical statics, three hours.—By Professor K. REISSNER: Mechanics, I and II, four hours; Aerodynamics, four hours.

BERLIN.—By Professor G. HETTNER: Differential and integral calculus, six hours; with exercises, two hours; Differential equations, two hours.—By Professor F. HAENTZSCHEL: Elements of the calculus, with exercises, six hours; Elements of mechanics, four hours.—By Professor O. KRIGAR-MENZEL: General mechanics, four hours; Advanced mechanics, four hours. —By Professor E. JOLLES: Descriptive geometry, I, six hours; Descriptive geometry, II, six hours.—By Professor E. LAMPE: Calculus, with exercises, six hours.; Definite integrals, four hours.—By Professor G. SCHEFFERS: Descriptive geometry, I and II, each six hours.—By Professor G. WALLENBERG: Selected chapters of elementary mathematics, two hours; Theory of functions, two hours; Theory of potential, two hours.—By Professor E. SALKOWSKY: Descriptive geometry, six hours.—By Dr. L. LICHTENSTEIN: Integral equations, six hours.—By Dr. R. FUCHS: Differential equations, two hours.—By Dr. K. v. IGNATOWSKY: Vector analysis, two hours.

BRESLAU.—By Professor C. CARATHÉODORY: Higher mathematics, four hours; Selected chapters in analysis, two hours; Mathematics for chemists and mining engineers, three hours.—By Professor G. HESSENBERG: Descriptive geometry, with exercises, eight hours; Selected chapters in geometry, three hours. —By Professor E. STEINITZ: Higher mathematics, III, with exercises, eight hours.

BRUNSWICK.—By Professor R. DEDEKIND: Elements of the theory of numbers, four hours; Introduction to the theory of probabilities.—By Professor R. FRICKE: Analytic geometry and algebra, five hours; Differential and integral calculus, four hours; Selected chapters in the theory of functions, three hours. —By Professor H. E. TIMERDING: Descriptive geometry, six hours; Introduction to higher analysis, four hours; Theory of surfaces and space curves, three hours.—By Dr. K. WITTE: Analytic mechanics, four hours.

DANZIG.—By Professor H. LORENZ: Dynamics of a rigid body, four hours: Elasticity, two hours.—By Professor H. v. MANGOLDT: Higher mathematics, five hours; Seminar, two hours.—By Professor F. SCHILLING: Descriptive geometry, with exercises, eight hours; Perspective and photogrammetry, one hour.—By Professor J. SOMMER: Higher mathematics, six hours; Seminar (with Professor v. Mangoldt) two hours.

DARMSTADT.—By Professor F. DINGELDEY: Higher mathematics, six hours; Theory of attraction, two hours.—By Professor F. GRAEFE: Repetitorium of elementary mathematics, three hours; Higher mathematics, five hours; History of mathematics, one hour.—By Professor J. HORN: Higher mathematics, I and II, each five hours.—By Professor R. MÜLLER: Descriptive geometry, eight hours.—By Professor H. WIENER: Descriptive geometry, eight hours; Foundations of elementary geometry, two hours.

DELFT.—By Professor J. A. BARRAU: Determinants and introduction to the calculus, three hours; Elements of projection two hours; Curved lines and surfaces, two hours; Line geometry, one hour.—By Professor W. H. L. JANSSEN v. RAAY: Advanced analytics and foundations of the calculus, five hours; Advanced calculus, four hours.—By Professor W. A. VERSLUYS: Plane analytic geometry, two hours; Geometry of space, two hours; Theory of equilibrium and of motion, two hours.—By Professor J. CARDINAAL: Descriptive geometry, two hours; Application to space curves and surfaces, two hours; Kinematics, three hours.—By Dr. G. SCHOUTEN: Geometry of motion, two hours; Moments of inertia and dynamics of a particle, two hours; Theory of the gyroscope, one hour.

DRESDEN.—By Professor G. HELM: Higher mathematics, five hours; Mathematical physics, three hours; Theory of the potential, three hours.—By Professor M. KRAUSE: Higher mathematics, five hours; Analytic functions, three hours;

Seminar, two hours.—By Professor L. LUDWIG: Descriptive geometry, six hours; Theory of perspective, two hours; Space curves, two hours.—By Professor E. NAETSCH: Analytic geometry of quadric surfaces, four hours; Elementary algebra and analysis, two hours.—By Dr. L. HEGER: Plane cubic curves, three hours.

HANOVER.—By Professor L. KIEPERT: Higher mathematics, I, eight hours; Higher mathematics, III, two hours; Calculus of variations, two hours.—By Professor F. MÜLLER: Higher mathematics, II, eight hours; Vector analysis, two hours.—By Professor K. WIEGHARDT: Foundations of higher mathematics, four hours; Elasticity, four hours; Differential equations, one hour.—By Professor K. RODENBURG: Descriptive geometry, nine hours.—By Dr. L. PETZOLD: Algebra and trigonometry, three hours.

KARLSRUHE.—By Professor M. DISTELI: Descriptive geometry, four hours; with exercises, four hours; Graphical statics, two hours; with exercises, two hours.—By Professor K. HEUN: Mechanics, with exercises, six hours; Seminar, three hours.—By Professor A. KRAZER: Higher mathematics, I (algebra and analytic geometry), with exercises, eight hours.—By Professor P. STÄCKEL: Higher mathematics, II (differential and integral calculus), three hours; Partial differential equations, two hours.—By Dr. H. MOHRMANN: Exercises in fundamental principles of mathematics, two hours; Arithmetic and algebra, with exercises, three hours; Trigonometry, with exercises three, hours.—By Dr. W. VOGT: Analytic geometry, with exercises, three hours; Projective geometry, three hours.—By ——: Elements of mechanics, with exercises, three hours.

STUTTGART.—By Professor R. MEHMKE: Descriptive geometry, six hours; Graphical methods, three hours.—By Professor L. KOMMERELL: Foundations of geometry, four hours.—By Professor E. WÖLFFING: Theory of functions, I, four hours. —By Professor G. FABER: Higher mathematics, II, six hours; Differential and integral calculus, with exercises, four hours.—By Dr. F. HALLER: Plane and spherical trigonometry, four hours.

PROFESSOR E. W. BROWN, of Yale University, has been elected an honorary fellow of Christ's College, Cambridge, England.

PROFESSOR F. LONDON, of the University of Bonn, has been promoted to a full professorship of mathematics.

PROFESSOR F. ENGEL, of the University of Greifswald, has been decorated with the order of the red eagle of the fourth class.

PROFESSOR E. STUDY, of the University of Bonn, has been elected to membership in the Göttingen academy of sciences.

PROFESSOR N. HATZIDAKIS, of the University of Athens, has been elected a member of the Vienna academy of sciences. Professor D. HILBERT, of the University of Göttingen, has been elected a corresponding member.

PROFESSOR PH. FURTWÄNGLER, of the agricultural academy of Poppelsdorf, has accepted a full professorship of mathematics at the University of Vienna.

DR. R. BÖHM has been appointed docent in mathematics at the University of Munich.

DR. J. L. JONES has been appointed instructor in mathematics at Yale University.

PROFESSOR L. G. WELD, until recently head of the department of mathematics and dean of the faculty of arts of Iowa State University, has been appointed director of the Pullman Free School of Manual Training to be established at Pullman, Ill.

PROFESSOR GEORGE W. JONES, of Cornell University, died October 29, at the age of 74 years. He retired as professor emeritus under the Carnegie foundation in 1907, after 45 years of teaching, the last 31 of which were spent at Cornell.

RECENT catalogues of second-hand books:—Heinrich Kerler, Ulm a. Donau, Antiquariats-Katalog 397, 2377 titles in pure and applied mathematics.—W. Heffer and Sons, Cambridge, England, catalogue 83, 2141 titles in mathematics, physics, and astronomy; also catalogue 81, 4507 titles of which 927 are school and college text-books in mathematics.—W. Junk, Kurfürstendamm 201, Berlin, bulletin 9, 2700 titles in exact sciences.—Ferdinand Schöningk, Osnabrück, catalogue 125, 1865 titles in mathematics and sciences.—J. Schweitzer, Lenbachplatz 1, Munich, catalogue 54, insurance, 2337 titles.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Breggen (J. v. d.). Vademecum der Wiskunde. Zutphen, 1911. 8vo. 81 pp. M. 1.00

Cailler (C.). Sur les notion de courbure et sur quelques points de géométrie infinitésimale non-euclidienne. Paris, Fischbacher, 1911. 4to. 60 pp. Fr. 5.00

Chambré (A.). Darstellung von Faktoren ganzer Funktionen durch Kovarianten. (Diss.) Giessen, 1910. 8vo. 36 pp.

Dahlhaus (G.) und Riemann (A.). Mathematisches Formelbuch. 2 Teile. Berlin, Günther, 1911. 107 pp. M. 4.00

Encyclopédie des sciences mathématiques pures et appliquées. Edition française. Tome III, 3me volume: Géométrie algébrique plane. Fascicule 1: Les coniques, par F. Dingeldey et E. Fabry. Leipzig, Teubner, 1911. 8vo. pp. 1–160. M. 6.00

Euler (L.). Opera omnia. Sub auspiciis Societatis Scientiarum Naturalium Helvetiae edenda curaverunt F. Rudio, A. Krazer et P. Stäckel. (45 Bände in 3 Serien.) Serie I, Band 1: Vollständige Anleitung zur Algebra, mit den Zusätzen von J. L. Lagrange. Herausgegeben von H. Weber, mit Vorwort zur Euler-Ausgabe und der Lobrede von N. Fuss. Leipzig, 1911. 4to. M. 28.50

Finzel (A.). Die Lehre vom Flächeninhalt in der allgemeinen Geometrie. (Diss.) Strassburg, 1911. 8vo. 46 pp.

Gambier (G.). Le mathématicien François Viète. Généalogie de sa famille. La Rochelle, Texier, 1911. 8vo. 31 pp.

International catalogue of scientific literature. Ninth annual issue. A: Mathematics. London, Harrison, 1911. 8vo. 228 pp. 15s.

Kiefer (A.). Die Einführung der homogenen Koordinaten durch K. W. Feuerbach. (Diss.) Strassburg, 1910. 8vo. 55 pp.

Kill (P.). Beiträge zum Fundamentalproblem der Flächentheorie. (Diss.) Strassburg, 1910. 8vo. 49 pp.

Lukjantschenko (S.). Integrirung der Differentialgleichungen. Mit Anhang: Doppelte und dreifache Integrale. 2te Auflage. (Russian.) Charkow, 1911. 8vo. 226+51 pp.

Mangoldt (H.). Einführung in die höhere Mathematik für Studierende und zum Selbststudium. 1ter Band: Anfangsgründe der Infinitesimalrechnung und der analytischen Geometrie. Leipzig, Hirzel, 1911. 8vo. 14+477 pp. M. 13.00

MacMahon (P. A.). Memoir on the theory of the partition of numbers. Part 6. (Royal Society.) London, Dulau, 1911. 4to. Sewed. 1s. 6d.

Mathematical questions and solutions from the Educational Times, new series. Volume 19. London, Hodgson, 1911. 8vo. 6s. 6d.

Milne (W. P.). Projective geometry for use in colleges and schools. London, Macmillan, 1911. 8vo. 164 pp. 2s. 6d.

Muir (T.). The theory of determinants in the historical order of development. Volume 2: The period 1841 to 1860. London, Macmillan, 1911. 8vo. 492 pp. 17s.

PICKERT (E.). Verallgemeinerung der Untersuchungen von Gauss über das arithmetisch-geometrische Mittel. (Diss.) Leipzig, 1911. 8vo. 67 pp.

RIEMANN (A.). See DAHLHAUS (G.).
SALTYKOV. Sur l'évolution de la théorie des équations partielles du premier ordre d'une fonction inconnue. (Russian.) St. Petersburg, 1911. 8vo.

SEEMAN (H.). Projektive Verallgemeinerung metrischer Begriffe. (Diss.) Giessen, 1910. 8vo. 24 pp.

THAER (F.). Analytische Beiträge zur Lehre vom Kegelschnittsystem (3 p, 1 l). (Diss.) Giessen, 1911. 28 pp.

VAERTING (M.). Zur Transformation der vielfachen Integrale. (Diss.) Giessen, 1910. 8vo. 24 pp.

WOLFF (G.). Ueber Kollineationen in der Ebene. (Diss.) Giessen, 1910. 8vo. 60 pp.

ZITSCHER (H.). Philosophische Untersuchungen über die Zahl. (Diss.) Greifswald, 1910. 8vo. 80 pp.

II. ELEMENTARY MATHEMATICS.

BÉCHÉ (A.). See DUSSAUX (E.).
BIOCHE (C.). Enseignement secondaire. (Rapports de la sous-commission française, volume II.) Paris, Hachette, 1911. 8vo. 157 pp. Fr. 4.00

BORCHARDT (W. G.) and PERROTT (A. D.). Geometry for schools. Volume 2 Stage 3. Section 1. London, Bell, 1911. 8vo. 118 pp. 1s. 6d.

BÖTTGER (A.). See THAER (A.).

BRANDENBERGER (K.). Der mathematische Unterricht an den Schweizerischen Gymnasien und Realschulen. Genève, Georg, 1911. 8vo. 163 pp. Fr. 3.50

BRYAN (G. H.) and DEAKIN (R.). The text-book of algebra. Part II: Senior. London, Clive, 1911. 8vo. 356 pp. 3s.

DEAKIN (R.). A text-book of algebra. Part I: Junior. London, Clive, 1911. 8vo. 372 pp. 3s.

——. See Bryan (G. H.).

DUSSAUX (E.) et BÉCHÉ (A.). Nouveau cours de géométrie. 3e année d'écoles primaires supérieures; préparation aux écoles nationales d'arts et métiers. Paris, Colin, 1911. 16mo. 391 pp. Fr. 3.00

DYCK (W. v.). Enseignement des sciences mathématiques, naturelles et techniques dans les écoles supérieures. Ixelles-Bruxelles, Bothy, 1911. 8vo. 67 pp.

GEUTHER (N.). See THAER (A.).

GROSSMANN (M.). Der mathematische Unterricht an der Eidgenössischen Technischen Hochschule. Genève, Georg, 1911. 52 pp. Fr. 2.00

HUN (J. G.) and MACINNES (C. R.). Logarithmic, trigonometric, and other tables. New York, Macmillan, 1911. 8vo. 2+104 pp. $0.60

——. The elements of plane and spherical trigonometry; with tables. New York, Macmillan, 1911. 8vo. 7+205 pp. $1.35

——. The elements of plane and spherical trigonometry; without tables. New York, Macmillan, 1911. 8vo. 7+101 pp. $0.90

Intermediate mathematics papers, from 1900 to 1911. (University Tutorial Series.) London, Clive, 1911. 8vo. 100 pp. Sewed. 2s.

Kurras (K.). Repetitorium des Rechenstoffes, der Arithmetik und Algebra. Berlin, Ulrich, 1911. 8vo. 32 pp. M. 1.30

Lietzmann (W.). See Schuster (M.).

Lübsen (H. B.). Ausführliches Lehrbuch der Arithmetik und Algebra zum Selbstunterricht und mit Rücksicht auf die Zwecke des praktischen Lebens. 27te Auflage. Neubearbeitet von A. Dornadt. Leipzig, Brandstetter, 1911. 8vo. 4+283 pp. Cloth. M. 4.50

MacInnes (C. R.). See Hun (J. G.).

Mahler (G.). Leitfaden für den Anfangsunterricht in der Algebra. 2te Auflage. Stuttgart, Bonz, 1911. 8vo. 6+127 pp. M. 1.50

Marsh (C. A.) and Phipps (H. J.). College entrance examination papers in plane geometry. New York, Merrill, 1911. 12mo. 178 pp. $0.60

Perrott (A.). See Borchardt (W. G.).

Phipps (H. J.). See Marsh (C. A.).

Renard (J.). La pédagogie à l'Université. Formation des professeurs d'athénée et spécialement des professeurs de mathématiques. Liège, Dessain, 1911. 8vo. 102 pp.

Schuster (M.). Geometrische Aufgaben und Lehrbuch der Geometrie nach konstruktiv-analytischer Methode. Nach dem Tode des Verfassers herausgegeben von W. Lietzmann. Ausgabe A: Für Vollanstalten. 2ter Teil: Trigonometrie. 2te, vermehrte und verbesserte Auflage. Leipzig, Teubner, 1911. 8vo. 8+118 pp. M. 1.80

Thaer (A.), Geuther (N.) und Böttger (A.). Der mathematische Unterricht in den Gymnasien und Realanstalten der Hansestädte, Mecklenburgs und Oldenburgs. (Abhandlungen über den mathematischen Unterricht in Deutschland. Veranlasst durch die internationale Unterrichtskommission. Herausgegeben von F. Klein. 1ter Band, 4tes Heft.) Leipzig, Teubner, 1911. 8vo. 6+93 pp. M. 2.00

Zuschlag (H.). Planimetrie. Für die Schüler der Quarta bis Oberprima und für Abiturienten. 2te Auflage. (Bibliothek Schüler-Versetzung. 7ter Band.) Berlin-Schoneberg, Mentor-Verlag, 1911. 112 pp. M. 1.00

——. Planimetrische Konstruktionsaufgaben (I). 2te Auflage. (Bibliothek Schüler-Versetzung. 8ter Band.) Berlin-Schöneberg, Mentor-Verlag, 1911. 69 pp. M. 1.00

——. Stereometrie. Anfängerkursus. 2te Auflage. (Bibliothek Schüler-Versetzung. 18ter Band.) Berlin-Schoneberg, Mentor-Verlag, 1911. 12+64 pp. M. 1.00

III. APPLIED MATHEMATICS.

Barton (E. H.). Analytical mechanics, comprising kinetics and statics of solids and fluids. London, Longmans, 1911. 8vo. 556 pp. 10s. 6d.

Bedell (F.) and Pierce (C. A.). Direct and alternating current manual; with directions for testing, and a discussion of the theory of electrical apparatus. 2nd edition, enlarged. New York, Van Nostrand, 1911. 8vo. 13+360 pp. $2.00

Blancarnoux (P.). Toute la mécanique rationnelle et appliqueé à la portée de tous. Panorama méthodique et complet en 12 volumes. Tome 3: Cinétique. Paris, Geisler, 1911. 8vo. 156 pp.

Boistel (E.). See Thompson (S. P.).

Broggi (H.). Versicherungsmathematik. Deutsche Ausgabe. Leipzig, Teubner, 1911. 8vo. 8+360 pp. Cloth. M. 8.00

Duhem (P.). Traité d'énergétique ou de thermodynamique générale. Tome 2: Dynamique générale. Conductibilité de la chaleur. Stabilité de l'équilibre. Paris, Gauthier-Villars, 1911. 8vo. 508 pp. Fr. 18.00

Ferguson (O. J.). The elements of electrical transmission; a text-book for colleges and technical schools. New York, Macmillan, 1911. 8vo. 7+457 pp. $3.50.

Geigel (R.). Die Wärme. (Bücher der Naturwissenschaft, herausgegeben von S. Günther.) Leipzig, Reclam, 1911. 16mo. 191 pp. M. 1.00

Hughes (H. J.) and Safford (A. T.). A treatise on hydraulics. London, Macmillan, 1911. 8vo. 14+505 pp. 16s.

King (W. R.). The elements of the mechanics of materials and of power transmission. New York, Wiley, 1911. 8vo. 5+266 pp. $2.50

Landré (C. L.). Mathematisch-technische Kapitel zur Lebensversicherung. 4te Auflage, verbessert und vermehrt von H. F. Landré. Jena, 1911. 8vo. 26+528 pp. M. 11.50

Martin (L.) Text-book of mechanics. Volume 3: Mechanics of materials New York, Wiley, 1911 12mo 229 pp. $1.50

Murdock (H. E.). Strength of materials. New York, Wiley 1911. 12mo. 14+308 pp. $2.00

Offley (C. N.). Engineering mechanics. Annapolis, United States Naval Institute, 1911. $3.50

Patterson (G W.). Revolving vectors with special application to alternating current phenomena. New York, Macmillan, 1911. 8vo. 6+89 pp. $1.00

Pender (H.). Principles of electrical engineering. New York, McGraw-Hill, 1911. 8vo. 18+438 pp. $4.00

Pierce (C. A.). See Bedell (F.).

Safford (A. T.). See Hughes (H. J.).

Smolik-Heller. Raumlehre und darstellende Geometrie fur die 4te bis 7te Klasse der Realschulen. Bearbeitet von K. Hahndel. 4te Auflage. Wien, Tempsky, 1911. K. 3.60

Spofford (C. M.). Theory of structures. New York, McGraw-Hill, 1911. 8vo. 432 pp. $4.00

Ssusslow (G. K.). Elements of analytical mechanics. (Russian.) Volume 1, Part 2: Dynamics of a particle. 2nd edition. Kiew, 1911. 8vo. 155 pp. $0.75

THOMPSON (S. P.). Traité théorique et pratique des machines dynamo-électriques. Traduit par E. Boistel. Courant continu. 4e édition française. Paris, Béranger, 1911. 8vo. 18+1060 pp.

TIMBIE (W. H.). Elements of electricity for technical students. New York, Wiley, 1911. 12mo. 11+556 pp. Cloth. $2.00

VITALE (A.). Trattato elementare di geometria descrittiva. Avellino, Pergola, 1911. 8vo. 65 pp.

VOGT (H.). Geometrie und Ökonomie der Bienenzelle. (Festschrift zur Jahrhundertfeier der Universität Breslau.) Breslau, Trewendt & Granier, 1911. 8vo. 63 pp. M. 3.00

WEGENER (A.). Thermodynamik der Atmosphäre. Leipzig, Barth, 1911. 8+331 pp. Cloth. M. 12.00

ZÜHLKE (P.). Der Unterricht im Linearzeichnen und in der darstellenden Geometrie an den deutschen Realanstalten. (Abhandlungen über den mathematischen Unterricht in Deutschland. 3ter Band, 3tes Heft.) Leipzig, Teubner, 1911. 8vo. 4+92 pp. M. 2.60

The Evanston Colloquium

Lectures on Mathematics

Delivered from Aug. 28 to Sept. 9, 1893, before members of the Congress of Mathematics held in connection with the World's Fair in Chicago at Northwestern University, Evanston, Ill.

By FELIX KLEIN

Reported by Alexander Ziwet. Republished by the American Mathematical Society, New York, 1911.

Professor Klein's Evanston Colloquium Lectures on Mathematics, which have for some years been out of print, have been republished by the American Mathematical Society and are now on sale at the nominal price of seventy-five cents per copy. The new edition is printed from the original plates, with correction of a few misprints. A brief preface by Professor W. F. Osgood has been added. The volumes are bound like the original. Orders should be addressed to the

AMERICAN MATHEMATICAL SOCIETY
501 West 116th Street, New York, N. Y.

Eisenhart: A Treatise on the Differential Geometry of Curves and Surfaces

By LUTHER P. EISENHART, Professor of Mathematics in Princeton University. 474 pages, with diagrams, $4.50

This Treatise introduces the student to the methods of differential geometry and to the theory of curves and surfaces developed thereby, thus preparing him for more extensive reading on the subject. A knowledge of the calculus, elementary differential equations, and the elements of coordinate geometry of three dimensions is presupposed. Hence the first half of the book may be used with seniors, and the remainder will constitute a full-year course for graduate students.

The method generally used is that of Gauss, common among German and Italian writers, but the kinematical method, frequently adopted in France, has been developed and applied where more feasible. Several hundred problems are included.

Hedrick: Goursat's Course in Mathematical Analysis—Volume I

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The French edition of this work attracted wide-spread attention, both because of the reputation of the author and on account of the clearness of its style, its wealth of material, and the rigor with which the subject matter was presented. The book is broad in scope. It commends itself to the undergraduate student in college because of its admirable clearness and simplicity, and thoroughly prepares for the study of the higher branches of analysis—notably, differential equations and the theory of functions.

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THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and fifty-fifth regular meeting of the Society was held in New York City on Saturday, October 28, extending through the usual morning and afternoon sessions. The attendance included the following thirty-five members: Professor W. J. Berry, Professor G. D. Birkhoff, Professor G. A. Bliss, Professor E. W. Brown, Professor B. H. Camp, Professor F. N. Cole, Dr. G. M. Conwell, Dr. H. B. Curtis, Dr. L. L. Dines, Professor L. P. Eisenhart, Professor H. B. Fine, Professor T. S. Fiske, Professor W. B. Fite, Professor W. A. Garrison, Professor C. C. Grove, Professor E. V. Huntington, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Dr. N. J. Lennes, Professor James Maclay, Dr. Emilie N. Martin, Dr. H. H. Mitchell, Dr. Anna J. Pell, Dr. H. W. Reddick, Professor R. G. D. Richardson, Miss S. F. Richardson, Professor Paul Saurel, Mr. L. P. Siceloff, Professor P. F. Smith, Professor H. D. Thompson, Dr. M. O. Tripp, Professor Oswald Veblen, Mr. H. E. Webb, Miss E. C. Williams.

President Fine occupied the chair at the two sessions. The Council announced the election of the following persons to membership in the Society: Professor T. B. Ashcraft, Colby College; Professor Clara L. Bacon, Goucher College; Professor J. M. Davis, State University of Kentucky; Professor W. C. Eells, Whitworth College; Dr. J. L. Jones, Yale University; Professor F. C. Kent, University of Oklahoma; Professor L. C. Plant, University of Montana; Mr. R. E. Powers, Denver, Colo.; Mr. T. M. Simpson, University of Wisconsin; Professor Evan Thomas, University of Vermont; Professor H. C. Wolff, University of Wisconsin; Mr. W. A. Zehring, Purdue University. Nine applications for membership in the Society were received.

A list of nominations for officers and other members of the Council, to be placed on the official ballot for the annual meeting, was adopted. Committees were appointed to make arrangements for the summer meeting of the Society at the University of Pennsylvania, in 1912, and to audit the Treasurer's accounts. The invitation of the University of Wisconsin to hold the summer meeting and colloquium at that University in 1913 was accepted.

A committee consisting of President Fine as chairman and Professors E. W. Brown, C. J. Keyser, E. H. Moore, W. F. Osgood, and E. B. Van Vleck was appointed to consider and report to the Council a plan for placing the business of the Society on a permanent basis. It was decided to change the form of the Annual Register by omitting in the personal entries all mention of membership in other organizations.

The following papers were read at this meeting:

(1) Mr. A. R. Schweitzer: "On a functional equation."

(2) Professor E. V. Huntington: "A new approach to the theory of relativity."

(3) Mr. L. P. Siceloff: "Simple groups from order 2,001 to order 3,640."

(4) Dr. H. H. Mitchell: "Determination of the quaternary linear groups by geometrical methods."

(5) Professor G. A. Bliss: "A new proof of the existence theorem for implicit functions."

(6) Mr. R. E. Powers: "The tenth perfect number."

(7) Professor E. W. Brown: "On the summation of a certain triply infinite series."

(8) Dr. L. L. Dines: "On the highest common factor of a system of polynomials."

(9) Professor R. D. Carmichael: "Fundamental properties of a reduced residue system mod m."

(10) Professor R. D. Carmichael: "A generalization of Cauchy's functional equation."

(11) Professor R. D. Carmichael: "On composite numbers P which satisfy the Fermat congruence $a^{P-1} \equiv 1 \bmod P$."

(12) Professor Edward Kasner: "Differential invariants of infinite order."

(13) Professor B. H. Camp: "Series of Laplace's functions."

(14) Dr. N. J. Lennes: "A new proof that a Jordan curve separates a plane."

In the absence of the authors the papers of Mr. Schweitzer, Mr. Powers, and Professor Carmichael were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Mr. Schweitzer proves the following theorem:

Let $u = \phi(x, z)$, $v = \phi(x, y)$ and $w = \phi(y, z)$; then if $\phi(u, v) = w$, there exists a function χ such that $\chi(v) = \chi(x) - \chi(y)$. A similar result is valid in case of the relation $\phi(u, w) = v$.

The method of proof is analogous to that of Abel * and Stäckel † who show (in effect) that if $\phi(y, u) = \phi(z, v) = \phi(x, w)$, then there exists a function χ such that $\chi(v) = \chi(x) + \chi(y)$.

2. The theory of relativity as developed by Einstein ‡ is usually supposed to involve a radical modification, not only of our conception of the ether, but also of our fundamental notions concerning space and time; and the discussion of the so-called "paradoxes of relativity" has often led beyond the safe ground of mathematical deduction into the realm of metaphysical speculation.

The purpose of Professor Huntington's article is to show that the famous "transformation equations," which stand at the center of the theory, can be easily deduced from simple conventions in regard to the setting of clocks and the laying out of coördinate systems, without any conflict with our ordinary conception of the ether, or with our ordinary notions of space and time, thereby freeing the theory from the least appearance of paradox.

The method pursued is in some sense a return to the point of view of Lorentz,§ who retained the concept of the ether as his starting point, and never abandoned our ordinary notions of time and space; and the transformation equations here obtained resemble the equations of Lorentz in being slightly more general than those of Einstein. Lorentz's method of deducing these equations, however, involves a large and difficult part of the modern electromagnetic theory, while the method here adopted depends only on the most elementary considerations. The present article is purely expository, and does not attempt any critical or historical discussion.||

3. In this paper Mr. Siceloff continues the search for new simple groups of low order. He finds that the only orders of simple groups between 2,001 and 3,640 are those already well-

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* *Crelle's Journal*, vol. 1 (1826), p. 11.

† *Zeitschrift für Mathematik und Physik*, vol. 42 (1897), p. 323.

‡ A. Einstein, "Zur Elektrodynamik bewegter Körper," *Annalen der Physik*, ser. 4, vol. 17 (1905), pp. 891–921.

§ H. A. Lorentz, Versuch einer Theorie der elektrischen und optischen Erscheinungen in bewegten Körpern, Leiden, 1895. Neudruck, Leipzig, 1906. Also, The Theory of Electrons (Columbia University lectures, 1906), Leipzig, 1909.

|| For an extensive bibliography of the theory of relativity, the reader is referred to an article by J. Laub, in the *Jahrbuch der Radioaktivität und Elektronik*, vol. 7 (1910), December, p. 405.

known, viz., 2,520 and 3,420. Whether there is a second simple group of order 2,520 has not been determined. The results are based chiefly on the study of the Sylow subgroups.

4. A complete enumeration of all primitive collineation groups in ordinary space of three dimensions has been given by Blichfeldt (*Mathematische Annalen*, 1905). This was done primarily by algebraic methods and involved the use of theorems concerning the irreducibility of equations containing roots of unity. It is the purpose of Dr. Mitchell's paper to determine these groups by geometrical methods and in such a way as to determine also the quaternary groups where the coefficients of the transformations are marks of the $GF(p^n)$, but the order of the group is not divisible by the prime modulus p. Properties of the linear complex are found to be of much service in the solution of the problem.

5. The paper of Professor Bliss has to do with the solution of a system of equations of the form

$$f_i(x_1, x_2, \cdots, x_m; y_1, y_2, \cdots, y_n) = 0 \quad (i = 1, 2, \cdots, n).$$

The theorem stating the existence of a set of solutions of the form

$$y_i = y_i(x_1, x_2, \cdots, x_m) \quad (i = 1, 2, \cdots, n)$$

for these equations is an old one, but in the well-known proofs of it a single equation is usually considered first, and then the proof is made for a system of equations by applying the principle of mathematical induction. In the present paper a proof is given which seems relatively simple, and which applies with equal convenience to either one or more equations.

6. The composition of Mersenne's numbers $2^p - 1$, p a prime, is known for all values of p not exceeding 100 with the exception of $p = 89$. E. Lucas, in volume 1 (1878) of the *American Journal of Mathematics*, proved the following theorem: "If $N = 2^{4q+1} - 1$, and if we form the series of residues (modulo N) 4, 14, 194, $\cdots$, each of which is equal to the square of the preceding, diminished by two units, the number N is prime if the first residue 0 lies between the $2q$th and the $(4q + 1)$th term." Mr. Powers shows that for $4q + 1 = 89$ the 88th term of the foregoing series is 0 (first 0); hence $2^{89} - 1$ is a prime number, and $2^{88}(2^{89} - 1)$ is the tenth perfect number.

7. In the expansion of Hill's infinite determinant there occurs a term which is equivalent to

$$\Sigma_i\Sigma_k\Sigma_{k'}\{i\}\{i+1\}\{i+k\}\{i+k+1\}\{i+k+k'\}\{i+k+k'+1\},$$

where each of the six factors is obtained by giving j the proper value in $1/\{j\}=\alpha^2-j^2$. Here α is unequal to any integer, i takes all integral values from $+\infty$ to $-\infty$, and k, k' all integral values from 2 to ∞. Hill gives the value with but slight indication of the method by which he obtained it. Professor Brown shows how it may be obtained without great labor, and is thus able to verify Hill's result.

8. In a paper before the Chicago meeting of the Society in April, Dr. Dines considered the conditions characterizing the existence of a common factor of first or higher degree of a system of n polynomials in one variable. It was found that, if a certain restriction be made upon the coefficients, then necessary and sufficient conditions can be stated in terms of the vanishing of $n-1$ determinants.

In the present paper, a matrix is constructed whose elements are the coefficients of the given polynomials arranged according to a simple rule, and which for $n=2$ reduces to the matrix of the well known Sylvester resultant of two polynomials. This matrix possesses the following properties:

(1) The vanishing of the matrix constitutes necessary and sufficient conditions for the existence of a common factor of first or higher degree. (A matrix of μ rows and ν columns, $\mu\leqq\nu$, is said to vanish when every determinant of order μ of the matrix vanishes.)

(2) The degree of the highest common factor of the system of polynomials can be stated in terms of the rank of the matrix.

(3) The coefficients of the highest common factor are determinants of the matrix, easily characterized.

(4) There are always $n-1$ determinants of the matrix whose vanishing is equivalent to the vanishing of the matrix.

9. The most important example of an abelian group is furnished by the $\varphi(m)$ integers of a reduced residue system mod m, the operation of combination being multiplication and reduction mod m. In the present paper Professor Carmichael determines the invariants of such a group and gives a method for constructing all groups of this class having the same in-

variants as a given group of the class. In this way all groups of the class, simply isomorphic with a given one, are determined. Let us define $\lambda(m)$ as the greatest exponent to which any element of the reduced residue system mod m belongs. A method is given for finding every x such that $\lambda(x) = \lambda(m)$, and incidentally for finding every x such that $\varphi(x) = \varphi(m)$. The paper closes with a table giving all possible solutions of $\lambda(x) = a$ for every possible value of a up to $a = 24$. The solutions are separated into sets according to the type of group defined by the corresponding reduced residue system. In each case the invariants of the group are given.

10. The nature of the generalization of Cauchy's functional equations effected in Professor Carmichael's second paper is indicated by the following theorem: If α is any positive or negative quantity and u_z is any function of z capable of assuming at least once each value in the set

$$2^{-s}n\alpha \quad (n, s = 0, 1, 2, \cdots;\ n \leqq 2^s),$$

then the most general function $f(x)$ which is continuous on the interval $(0, \alpha)$ and satisfies the functional equation

$$f(x + u_z) = f(x) + f(u_z)$$

is $f(x) = ax$, where a is an arbitrary constant.

11. In his note on the Fermat congruence $a^{P-1} \equiv 1 \bmod P$ Professor Carmichael gives a method of finding composite numbers P which satisfy the congruence for every a which is prime to P.

12. When an analytic curve is referred to the tangent and normal at one of its points, its equation takes the form

$$y = a_2x^2 + a_3x^3 + \cdots.$$

The coefficients are differential invariants of the curve, the first being half the curvature. Every combination of the coefficients is an invariant, that is, intrinsically related to the curve. Professor Kasner considers combinations involving an infinitude of coefficients. An important example is the radius of convergence of series y. Another invariant is the quantity J defined as the first non-vanishing term in the sequence a_2, a_3, $\cdots$. This vanishes only when the curve is a straight line, and is applied to certain questions concerning geodesics and trajectories. The even and odd components of a curve at a given point, introduced by the author in an earlier paper, are examples of covariants of infinite order.

13. The most important theorem concerning the validity of the expansion of an arbitrary function in a series of Laplace's functions has been proved by Jordan in his Cours d'Analyse, second edition, volume 2, page 252. The conditions stated there are that the given function be continuous on the surface of the sphere within some small circle about the point at which the expansion is made, and that it have limited variation along every great circle through the point.

The object of Professor Camp's paper is to correct an error in Jordan's theorem, and to furnish new sufficient conditions for the validity of these expansions. To the conditions announced by Jordan should be added the requirements that the values of the variations be all less than some fixed number, and that these variations be uniform with respect to all great circles through the point. Furthermore, the conditions of continuity of the function and of uniformity of the variations may be replaced by the different condition that there exist a small circle about the point considered such that within it the given function is, along all great circles through the point, an indefinite integral of another function which has an absolutely convergent double Lebesgue integral in its domain of definition. Any or all of these conditions may fail on a null set of great circles through the point.

14. In a paper presented to the Society at the April meeting Dr. Lennes proved that a closed continuous surface separates a three-space in which it lies into two connected sets. The present paper is an adaptation of the methods used in that paper to the case of the curve in the plane.

F. N. COLE,
Secretary.

THE OCTOBER MEETING OF THE SAN FRANCISCO SECTION.

THE twentieth regular meeting of the San Francisco Section of the American Mathematical Society was held at the University of California on Saturday, October 28, 1911. The following seventeen members of the society were present:

Professor R. E. Allardice, Mr. B. A. Bernstein, Professor H. F. Blichfeldt, Dr. Thomas Buck, Professor G. C. Edwards, Professor M. W. Haskell, Professor L. M. Hoskins, Dr. Frank Irwin, Mr. C. G. P. Kuschke, Professor A. O. Leuschner, Professor J. H. McDonald, Professor W. A. Manning, Professor H. C. Moreño, Professor C. A. Noble, Professor E. W. Ponzer, Professor T. M. Putnam, Professor A. W. Whitney.

In the absence of Professor Lehmer, chairman of the section, Professor Haskell presided over the morning session and Professor Allardice over the afternoon session.

The following officers were elected for the ensuing year: president, Professor L. M. Hoskins; secretary, Professor T. M. Putnam; program committee, Professors Blichfeldt, McDonald, and Putnam.

The dates for the meetings for 1912 were fixed at April 6 and October 26.

The following papers were presented:

(1) Professor W. A. MANNING: "The primitive groups of class twelve."

(2) Professor G. A. MILLER: "Note on the maximal cyclic subgroups of a group of order p^m."

(3) Dr. H. W. STAGER: "A Sylow factor table of the first twelve thousand numbers."

(4) Professor A. W. WHITNEY: "Theory of schedule rating in fire insurance."

(5) Professor H. F. BLICHFELDT: "On a basis of plane geometry" (preliminary communication).

(6) Professor A. O. LEUSCHNER and Mr. B. A. BERNSTEIN: "Graphical solutions of the fundamental equations in the short methods of determining orbits."

In the absence of the authors the papers of Professor Miller and Dr. Stager were read by title. Abstracts of the papers follow below.

1. In addition to the 25 known primitive groups of class 12 on not more than 17 letters, Professor Manning has found that there are four others, one of degree 27, two of degree 28, and one of degree 36. Of these, only one is doubly transitive. It, and its maximal subgroup, which is primitive of degree 27 and of order $27 \cdot 1920$, seem not to have been noticed before.

2. It is known that every subgroup of order p^α in a group G of order p^m, $m > \alpha$ and p a prime number, is transformed into itself by at least $p^{\alpha+1}$ operators of G. Professor Miller calls a cyclic subgroup which is not contained in a larger cyclic subgroup of G a maximal cyclic subgroup of G, and proves the following related theorems:

A necessary and sufficient condition that every maximal cyclic subgroup of order p^α in a group G of order p^m, $m > 3$, is transformed into itself by no more than $p^{\alpha+1}$ operators of G is that G contains one and only one cyclic subgroup of order p^{m-1}. Every non-cyclic group of order p^m, with the exception of the three of order 2^m which involve one and only one cyclic subgroup of order 2^{m-1}, contains at least one maximal cyclic subgroup of order p^α which is transformed into itself by more than $p^{\alpha+1}$ operators of the group.

3. The main purpose of the table described in detail in Dr. Stager's paper is to furnish directly information as to the possible number of subgroups of a group of given order under Sylow's well known theorem. Each number is expressed as the product of powers of primes, and for each prime factor greater than 2 there is given the values of k, other than zero, of all divisors of the number of the form $p(kp + 1)$. Those values of k, other than zero, such that the number is identically equal to $p(kp + 1)$ are specifically indicated. The table is so arranged that all of the above information appears for any number as one entry. In the case of primes, their rank is indicated by the symbol p_n. The desired possible number of subgroups is obtained by substituting the listed values of p and k in the formula $kp + 1$.

In addition, a list of those numbers which contain no factors of the form $p(kp + 1)$, $k > 0$, is given, so arranged that the number of such numbers between any two limits less than 12,000 is easily obtained. The table was constructed independently by two different methods and the results compared for errors. The one method used is new and was probably applied for the first time in the construction of this table.

4. Since in fire insurance it is impossible to restrict the selection of risks to those that are standard, as is for the most part the case in life insurance, it becomes necessary to make an analysis of the hazard. A rate built up in this way is called a schedule rate. Mathematically the rate is a function of the various elements of construction, exposure, occupancy, etc., and the schedule is the expression of this relation. Existing schedules, while formed upon no real mathematical analysis, are as a matter of fact in the form of a Maclaurin expansion; a base rate is assumed and the departures from this are produced by additions. While this method works fairly well within a restricted field, it breaks down entirely, as might be expected, when the schedule is extended to a great variety of risks, and the consequence is that a great number of different schedules must be employed.

In the present paper Professor Whitney attempts by means of simplifying assumptions to make an analysis of the structure of this function. The result is a schedule no more complicated but radically different in structure. It is believed that a schedule of this form possesses a flexibility that makes it capable of universal application. The paper was originally given before the Actuarial Society of America and was published in the *Transactions* of that Society for May, 1911; rewritten in unmathematical form, it was published in the *Weekly Underwriter* of September 30.

5. In a paper published in the *Transactions* for 1902, pages 467–481, Professor Blichfeldt gave certain postulates which were sufficient to determine, for space of n dimensions, the typical forms for the distance between two points. In the present paper it is shown how for $n = 2$ a simpler set of postulates can be substituted. The existence of distance relations for four points is assumed only if one or two of the four points are one or both of two fixed points in the "plane." It is not assumed that the number of points of the plane is infinite.

6. In the preliminary determination of an orbit, the geocentric distance of the body can be obtained by solving the equation

$$(z - p')^2 + q'^2 - \frac{h}{[(z - c)^2 + s^2]^{\frac{3}{2}}} = 0 \tag{1}$$

if the orbit is assumed to be a parabola, and by solving

$$(z^2 - 2cz + 1)^3(z - m)^3 - m^2 = 0 \tag{2}$$

if no assumption is made regarding the orbit. One of the chief aims of Professor Leuschner's short methods for computing an orbit is to shorten the time taken for the calculation. The semi-graphical solution of the above equations used in the short methods contributes largely to this end, as well as to the lessening of tedious computational work; but this method still requires the use of tables. It is to eliminate entirely all computations in the solution of equation (1) above that Professor Leuschner and Mr. Bernstein devised methods of constructing geometrically the curve given by the equation

$$y = \frac{h}{[(z - c)^2 + s^2]^{\frac{3}{2}}},$$

whose intersections with the parabola $y = (z - p')^2$ give the required roots. By finding a geometric construction of the curve

$$z - m = \frac{m}{(z^2 - 2c + 1)^{\frac{3}{2}}},$$

whose intersections with the line $my = x + c - m$ give the roots of equation (2), Mr. Bernstein derived a purely graphical solution of the problem for the general orbit.

T. M. PUTNAM,
Secretary of the Section.

THE CARLSRUHE MEETING OF THE GERMAN MATHEMATICAL SOCIETY.

THE annual meeting of the Deutsche Mathematiker-Vereinigung was held in affiliation with the eighty-third convention of the society of naturalists and physicians at Karlsruhe during the week of September 24–28, under the presidency of Professor F. Schur, of the University of Strassburg. Under the stimulus of the pleasure and profit derived from preceding meetings a new record in attendance was established, over one hundred persons being present. The people of Karlsruhe generously provided for the social part of the meeting by arranging an excellent program for every evening, one being devoted to an excursion to Heidelberg to witness the illumination of the castle, and to hear a well-rendered concert. The general session of Monday morning was opened with an address of welcome by the Grand Duke of Baden. Sectional meetings were held during the next four days, and the business meeting on Thursday morning. The membership of the society is 765 and the reserve fund consists of over M. 23,000. The bibliographic commission has been discontinued, and new members have been added to the committee to provide for the further publication of the papers of E. Schröder. Two finished volumes of the works of Euler were presented, and a detailed report of the German subcommittee of the international commission of mathematical instruction was read. Professors Rohn and Sommerfeld succeed Professors Engel and Schur as members of the council. The official representatives to the fifth congress of mathematicians are the new chairman, Professor W. von Dyck, and the secretary, Professor M. Krazer. The next meeting of the Vereinigung will be held at Münster in September, 1912, under the presidency of Professor von Dyck.*

The following papers were read at this meeting.

(1) Professor C. Carathéodory, Breslau: "On the depiction of the boundary in the conformal representation of a general region."

(2) Professor O. Blumenthal, Aachen: "Meromorphic functions of several variables."

(3) Professor G. Faber, Stuttgart: "On interpolation."

(4) Professor L. G. Du Pasquier, Neuenburg: "On complex tettarions."

(5) Dr. L. Lichtenstein, Berlin: "On Poisson's integral and the behavior of analytic functions on the boundary of the circle of convergence."

(6) Professor L. Schlesinger, Giessen: "On early work of Gauss on the arithmetic-geometric mean."

(7) Professor K. Heun, Karlsruhe: "Outlines for the extension of classical mechanics" (report).

(8) Professor P. Voronetz, Kieff: "On the equations of motion of a rigid body."

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* For this part of the report the editors are indebted to Dr. H. Mohrmann, of Karlsruhe; for that of the social activities, to Dr. Clara A. Smith. The abstracts of the paper were kindly furnished by the authors in reply to a request sent by the editors.

(9) Professor R. v. MISES, Strassburg: "On the problems of oscillation and the Sommerfeld theorem of turbulence."

(10) Professor G. HAMEL, Brünn: "On the stiffness of ropes."

(11) Professor V. VARICAK, Agram: "On the non-euclidean interpretation of the theory of relativity."

(12) Professor E. PAPPERITZ, Freiberg: "Exhibition of the completed kinodiaphragmatic projection apparatus for geometric illustrations."

(13) Professor H. WIENER, Darmstadt: "Determination of a root of an equation by means of a convergent (or divergent) series."

(14) Professor E. SALKOWSKI, Charlottenburg: "Theory of curves in elliptic space."

(15) Professor C. JUEL, Copenhagen: "Concerning the straight lines which lie on an analytic non-algebraic surface of order three."

(16) Professors F. KLEIN, Göttingen, E. BROUWER, Amsterdam, and P. KOEBE, Leipzig: "Report on automorphic functions and the problem of uniformization."

(17) Professor E. HILB, Würzburg: "Connection between fundamental regions and linear differential equations."

(18) Dr. L. BIEBERBACH, Königsberg: "Remarks on automorphic functions."

(19) Professor J. WELLSTEIN, Strassburg: "On motion in space and the principle of relativity."

(20) Professor L. HEFFTER, Freiburg: "On the introduction of the four-dimensional world of Minkowski."

(21) Professor F. BERNSTEIN, Göttingen: "On Lagrange's problem of mean motion in the theory of secular perturbations."

(22) Professor H. WIENER, Darmstadt: "Exhibition and explanation of some new models."

Abstracts of several of the papers follow; the numbers correspond to those in the list above.

2. As an extension of memoirs by E. E. Levi, Professor Blumenthal discussed the natural boundaries of functions of two complex variables. He showed by means of the principle of condensation that the natural boundaries approach certain three-dimensional manifolds, many of the properties of which can be ascertained. The memoir is to appear in the *Jahresbericht*.

3. Professor Faber spoke of his investigations concerning

the accuracy of the best formulas of interpolation, and pointed out that they were less applicable to the representation of a function than to obtaining numerical approximations in mechanical quadrature. At the conclusion of the discussion the methods of Tchèbichef and of Cauchy were compared.

4. The tettarions discussed by Professor Du Pasquier are defined as matrices of numbers having the same number of rows as of columns. They are added or subtracted by adding or subtracting corresponding components. Multiplication follows the same laws as the composition of linear substitutions. As subgroups, the complex numbers of Gauss and the quaternions of Hamilton are included. If $\nu = 4$ and all the components are real rational numbers, a tettarion field is defined. Division into integral and fractional forms can be made in an unlimited number of ways. Thus an infinite number of realms of rationality exist. A rational basis of classification is into integral, conditionally fractional, unconditionally fractional, and finally forms having both characters.

5. In the paper of Dr. Lichtenstein a number of properties of the partial derivatives of the Poisson integral, taken on the circumference, are derived. It is then shown that if $u(x, y)$ is a Poisson integral with integrable boundary function and $v(x, y)$ its complementary potential function, then the latter is integrable. The most important result concerns the partial derivatives of the second order of the logarithmic potential

$$u(x, y) = \frac{1}{4\pi}\int\int_T \log\{(x-\xi)^2 + (y-\eta)^2\}\, P(\xi, \eta)\, d\xi d\eta$$

in which $P(\xi, \eta)$ is an integrable function and T any finite region. It is shown that except for a set of points of content zero, the derivatives $\partial^2 u/\partial x^2$, $\partial^2 u/\partial y^2$ exist, are integrable, and satisfy the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = P(x, y).$$

8. The differential equations of motion of a rigid body are usually referred to a fixed system $O_1\text{-}x_1y_1z_1$ of rectangular coordinates, or to a system $O\text{-}xyz$ moving rigidly with the body. In the few cases in which a system $M\text{-}\xi\eta\zeta$ is employed which is movable as to both, M is either supposed fixed, or is taken as the center of gravity of the moving body.

In Professor Voronetz's paper, the first part is concerned with the equations of motion in M-$\xi\eta\zeta$ coordinates, the position of M being unrestricted; the second part develops the case of rolling motion further, and contains a number of new results.

11. Professor Varicak showed in his paper that the Lobachevsky geometry furnishes adequate means for the treatment of the theory of relativity; all of the investigations operate in a Lobachevsky space. By means of simple considerations, the addition theory for velocities is developed, as well as the laws of aberration, of reflection in moving mirrors, and certain other laws.

12. Professor Papperitz presented an improved complete form of his apparatus for geometric illustrations. For the principle of the procedure and its applicability for producing figures in space for the purposes of instruction, reference may be made to two short reports ("Ueber das Zeichnen in Raume," *Jahresbericht,* volume 19 (1911), page 307, and "Die kinodiaphragmatische Projection . . . ," *Hoffmann's Zeitschrift,* volume 11 (1911), page 465). This method of illustration promises to be particularly expedient; it represents not only figures in a plane, but also stereometric ones by means of virtual images of space curves and surfaces. The apparatus has been patented, and is provided with a number of models for representation. A detailed prospectus is furnished with the machine.

14. The principle of correspondence introduced by Study (*American Journal,* volume 29, pages 116 ff.) by means of which the speers of elliptic space are associated with the point-spheres of two spheres of unit radius in euclidean space is applied by Professor Salkowski to the theory of space curves. If x_{l_1} x_{l_2}, x_{l_3}, x_{r_1}, x_{r_2}, x_{r_3} are the speer coordinates of the directed tangent of a space curve, y_{l_i}, y_{r_i}; z_{l_i}, z_{r_i} ($i = 1, 2, 3$) those of its principal normal and binormal respectively, κ, τ the curvature and torsion, then the equations exist

$$\frac{dx_{l_i}}{ds} = \kappa y_{l_i}, \quad \frac{dx_{r_i}}{ds} = \kappa y_{r_i},$$

$$\frac{dy_{l_i}}{ds} = -\kappa x_{l_i} - (\tau - 1)\, x_{l_i}, \quad \frac{dy_{r_i}}{ds} = -\kappa x_{r_i} - (\tau + 1) z_{r_i},$$

$$\frac{dz_{l_i}}{ds} = (\tau - 1)\, y_{r_i}, \quad \frac{dz_{r_i}}{ds} = (\tau + 1) y_{r_i}.$$

These equations determine a curve of elliptic space, and, except as to a simple transformation, associate two curves c_l, c_r of euclidean space. A number of consequences of this theorem were pointed out.

16. The report on automorphic functions was presented by Professor Koebe. The most emphasis was put on the problem of uniformization, and a statement of recent progress made was given. After discussing his own publications (*Jahresbericht*, for 1906, 1907, 1910; *Göttinger Nachrichten*, 1907–1911; *Mathematische Annalen*, volumes 67, 69; *Crelle's Journal*, volumes 138, 139), the speaker pointed out the significance of recent memoirs of Poincaré, Hilbert, and others.

17. If four singular (real) points and their corresponding exponents of a linear differential equation of the second order are known, there are ∞^1 real values of the parameter, so that the circular quadrilateral has an orthogonal circle. In case of complex parameters, an infinite number of new theorems can be stated, one associated with each characteristic oscillation number. To generalize the above conditions, Professor Hilb uses the relations established by Klein for the triangle and by Ihlenberg for the quadrilateral. It is shown that the method developed by the latter becomes too complicated to be further extended. The author showed how to pass from polygons of n sides to those of $n + 1$ sides by a new method. The corresponding extension in the complex field is much more complicated, but can be made. By means of the Fricke normal polygons, it is possible to overcome the difficulty imposed by the Stieltjes boundary, making the transformation of y' (or higher derivative) for y, then taking the auxiliary points into account.

18. Dr. Bieberbach reported on the contents of the third installment of Fricke-Klein, Vorlesungen über automorphe Funktionen, and made some remarks concerning the proof by means of continuity of the theorem of the closed section.

VIRGIL SNYDER.

A NEW PROOF OF THE EXISTENCE THEOREM FOR IMPLICIT FUNCTIONS.

BY PROFESSOR GILBERT AMES BLISS.

(Read before the American Mathematical Society October 28, 1911.)

THE theorem with which this paper has to do is the one which states the existence of a set of functions

$$y_i = y_i(x_1, x_2, \cdots, x_m) \qquad (i = 1, 2, \cdots, n)$$

which satisfy a system of equations of the form

$$(1) \qquad f_i(x_1, x_2, \cdots, x_m; y_1, y_2, \cdots, y_n) = 0 \quad (i = 1, 2, \cdots, n).$$

For the case in which the functions f are only assumed to be continuous and to have continuous first derivatives, the proof seems to have been originally given by Dini.* His method is to show the existence of a solution of a single equation, and then to extend his result by mathematical induction to a system of the form given above, a plan which has been followed, with only slight alterations and improvements in form, by most writers on the theory of functions of a real variable. In a more recent paper† Goursat has applied a method of successive approximation which enabled him to do away with the assumption of the existence of the derivatives of the functions f with respect to the independent variables x.

One can hardly be dissatisfied with either of these methods of attack. It is true that when the theorem is stated as precisely as in the following paragraphs, the determination of the neighborhoods at the stage when the induction must be made is rather inelegant, but the difficulties encountered are not serious. The introduction of successive approximations is an interesting step, though it does not simplify the situation and indeed does not add generality with regard to the assumptions on the functions f. The method of Dini can in fact, by only a slight modification, be made to apply to cases where the functions do not have derivatives with respect to the variables x.

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* Lezioni di Analisi infinitesimale, vol. 1, chap. 13. For historical remarks, see Osgood, Encyclopädie der mathematischen Wissenschaften, II, B 1, § 44 and footnote 30.

† *Bulletin de la Société mathématique*, vol. 31 (1903), page 185.

The proof which is given in the following paragraphs seems to have advantages in the matter of simplicity over either of the others. It applies equally well, without induction, to one or a system of equations, and requires only the initial assumptions which Goursat mentions in his paper.

Where it is possible without sacrificing clearness, the row letters f, x, y, p, a, b will be used to denote the systems

$$\begin{array}{ll} f = (f_1, f_2, \cdots, f_n), & x = (x_1, x_2, \cdots, x_m), \\ y = (y_1, y_2, \cdots, y_n), & a = (a_1, a_2, \cdots, a_m), \\ b = (b_1, b_2, \cdots, b_n), & p = (a_1, a_2, \cdots, a_m; b_1, b_2, \cdots, b_n). \end{array}$$

In this notation the equations (1) have the form

$$f(x; y) = 0,$$

the interpretation being that every element of f is a function of $x_1, x_2, \cdots, x_m$; $y_1, y_2, \cdots, y_n$, and every f_i is to be set equal to zero. The notations p_ϵ, a_ϵ, b_ϵ represent respectively the neighborhoods

$$| x - a | < \epsilon, \; | y - b | < \epsilon; \quad | x - a | < \epsilon; \quad | y - b | < \epsilon$$

of the points p, a, b.

With these notations in mind the fundamental theorem which is to be proved may be stated as follows:

Hypotheses:

1) *the functions* $f(x; y)$ *are continuous, and have first partial derivatives with respect to the variables* y *which are also continuous, in a neighborhood of the point* p;

2) $f(a; b) = 0$;

3) *the functional determinant* $D = \partial(f_1, f_2, \cdots, f_n)/\partial(y_1, y_2, \cdots, y_n)$ *is different from zero at* p.

Conclusions:

1) *a neighborhood* p_ϵ *can be found in which there corresponds to a given value* x *at most one solution* $(x; y)$ *of the equations* $f(x; y) = 0$;

2) *for any neighborhood* p_ϵ *with the property just described a constant* $\delta \leqq \epsilon$ *can be found such that every* x *in* a_δ *has associated with it a point* $(x; y)$ *which satisfies the equations* $f(x; y) = 0$;

3) *the functions* $y(x_1, x_2, \cdots, x_m)$ *so found are continuous in the region* a_δ.

For the neighborhood p_ϵ let one be chosen in which the continuity properties of the functions f are preserved. If $(x; y)$ and $(x; y')$ are two points in p_ϵ, it follows by applying Taylor's formula to the differences $f(x; y') - f(x; y)$ that

$$f_1(x; y') - f_1(x; y) = \frac{\partial f_1}{\partial y_1}(y_1' - y_1) + \cdots + \frac{\partial f_1}{\partial y_n}(y_n' - y_n),$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$f_n(x; y') - f_n(x; y) = \frac{\partial f_n}{\partial y_1}(y_1' - y_1) + \cdots + \frac{\partial f_n}{\partial y_n}(y_n' - y_n),$$

where the arguments of the derivatives $\partial f_i/\partial y_k$ have the form $x; y + \theta_i(y' - y)$ and $0 < \theta_i < 1$. The determinant of these derivatives is different from zero when $(x; y') = (x; y) = (a; b)$, and hence must remain different from zero if p_ϵ is restricted so that in it the functional determinant D remains different from zero. It is then impossible that $(x; y)$ and $(x; y')$ should both be solutions of the equations $f(x; y) = 0$ if y is distinct from y'.

In the corresponding region b_ϵ the function

$$\varphi(a; y) = f_1^2(a; y) + f_2^2(a; y) + \cdots + f_n^2(a; y)$$

has a minimum for $y = b$, since for that value it vanishes and for every other it is positive. In particular

$$\varphi(a; \eta) - \varphi(a; b) > m > 0$$

for the *closed* set of points η forming the boundary of b_ϵ, on account of the continuity of φ, and the inequality

$$\varphi(x; \eta) - \varphi(x; b) > m$$

remains true for all values x in a suitably chosen domain a_δ. Hence for a fixed x in a_δ the minimum of $\varphi(x; y)$ is attained at a point y *interior* to b_ϵ. At such a point, however,

$$\frac{1}{2}\frac{\partial \varphi}{\partial y_1} = f_1\frac{\partial f_1}{\partial y_1} + f_2\frac{\partial f_2}{\partial y_1} + \cdots + f_n\frac{\partial f_n}{\partial y_1} = 0,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\frac{1}{2}\frac{\partial \varphi}{\partial y_n} = f_1\frac{\partial f_1}{\partial y_n} + f_2\frac{\partial f_2}{\partial y_n} + \cdots + f_n\frac{\partial f_n}{\partial y_n} = 0,$$

and this can happen only when all the elements of f are zero, since the functional determinant D is different from zero in p_ϵ. It follows that to every point x in a_δ there corresponds in p_ϵ a solution $(x; y)$ of the equations $f(x; y) = 0$.

The functions $y(x_1, x_2, \cdots, x_m)$ defined in this way over the region a_δ are all continuous. For consider the values y and $y + \Delta y$ corresponding to two points x and $x + \Delta x$. By applying Taylor's formula it follows from the relations

$$f(x; y + \Delta y) - f(x; y) = f(x; y + \Delta y) - f(x + \Delta x; y + \Delta y),$$

which are true because $(x; y)$ and $(x + \Delta x; y + \Delta y)$ both make $f = 0$, that

$$
\begin{aligned}
&\frac{\partial f_1}{\partial y_1}\Delta y_1 + \frac{\partial f_1}{\partial y_2}\Delta y_2 + \cdots + \frac{\partial f_1}{\partial y_n}\Delta y_n \\
&\qquad\qquad = f_1(x; y + \Delta y) - f_1(x + \Delta x; y + \Delta y), \\
(2)\quad &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
&\frac{\partial f_n}{\partial y_1}\Delta y_1 + \frac{\partial f_n}{\partial y_2}\Delta y_2 + \cdots + \frac{\partial f_n}{\partial y_n}\Delta y_n \\
&\qquad\qquad = f_n(x; y + \Delta y) - f_n(x + \Delta x; y + \Delta y),
\end{aligned}
$$

where the arguments of the derivatives $\partial f_i / \partial y_k$ have the form $x; y + \theta_i \Delta y$ $(0 < \theta_i < 1)$. The determinant of these derivatives is different from zero on account of the way in which p_ϵ was chosen, and the second members of the equations approach zero with Δx. Hence the same must be true of the quantities Δy, and the functions $y(x_1, x_2, \cdots, x_m)$ are seen to be continuous.

A similar application of Taylor's formula leads to the conclusion that

If the functions f have derivatives of the first order with respect to x_k which are continuous in the neighborhood of p, so have also the functions $y(x_1, x_2, \cdots, x_m)$ in the region a_δ; and if the f's have all derivatives of the nth order continuous, so have the functions $y(x_1, x_2, \cdots, x_m)$.

For suppose

$$\Delta x_1 \neq 0, \quad \Delta x_2 = \Delta x_3 = \cdots = \Delta x_m = 0.$$

Then by applying Taylor's formula to the second members of equations (2) it follows that

$$\frac{\partial f_1}{\partial y_1}\frac{\Delta y_1}{\Delta x_1}+\frac{\partial f_1}{\partial y_2}\frac{\Delta y_2}{\Delta x_1}+\cdots+\frac{\partial f_1}{\partial y_n}\frac{\Delta y_n}{\Delta x_1}+\frac{\partial f_1}{\partial x_1}=0,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\frac{\partial f_n}{\partial y_1}\frac{\Delta y_1}{\Delta x_1}+\frac{\partial f_n}{\partial y_2}\frac{\Delta y_2}{\Delta x_1}+\cdots+\frac{\partial f_n}{\partial y_n}\frac{\Delta y_n}{\Delta x_1}+\frac{\partial f_n}{\partial x_1}=0,$$

where the arguments of the derivatives $\partial f_i/\partial x_1$ have the form $x+\theta_i'\Delta x;\ y+\Delta y$. Hence as Δx_1 approaches zero the quotients $\Delta y_i/\Delta x_1$ approach limits $\partial y_i/\partial x_1$ which satisfy the equations

$$\begin{array}{ll} & \frac{\partial f_1}{\partial y_1}\frac{\partial y_1}{\partial x_1}+\frac{\partial f_1}{\partial y_2}\frac{\partial y_2}{\partial x_1}+\cdots+\frac{\partial f_1}{\partial y_n}\frac{\partial y_n}{\partial x_1}+\frac{\partial f_1}{\partial x_1}=0,\\ (3) & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & \frac{\partial f_n}{\partial y_1}\frac{\partial y_1}{\partial x_1}+\frac{\partial f_n}{\partial y_2}\frac{\partial y_2}{\partial x_1}+\cdots+\frac{\partial f_n}{\partial y_n}\frac{\partial y_n}{\partial x_1}+\frac{\partial f_n}{\partial x_1}=0,\end{array}$$

where the arguments of the derivatives of f are now $(x;\ y)$. A similar consideration shows the existence of the first derivatives with respect to the variables $x_2,\ x_3,\ \cdots,\ x_n$. The existence of the higher derivatives follows from the observation that the solutions of equations (3) are differentiable $n-1$ times with respect to the variables x on account of the assumption that the functions f are differentiable n times.

ON A SET OF KERNELS WHOSE DETERMINANTS FORM A STURMIAN SEQUENCE.

BY MR. H. BATEMAN, M.A.

WEYL * has recently given a theorem which states that if a kernel

$$k_n(s,t)=\sum_{p,q=1}^{n}k_{pq}\Phi_p(s)\Phi_q(t) \qquad (k_{pq}=k_{qp})$$

is formed from n functions $\Phi_p(s)$ whose squares are integrable in the interval (0, 1), then the smallest positive root of the

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* *Göttinger Nachrichten*, 1911, Heft 2, p. 110.

kernel

$$h_n(s, t) = k(s, t) - k_n(s, t)$$

is not greater than the $(n+1)$th positive root of $k(s, t)$.

It has occurred to me that this theorem is a particular case of the following general theorem:

Let $k(s, t)$ *be a symmetric function such that*

$$\int_0^1 \int_0^1 [k(s, t)]^2 ds dt$$

is convergent, $a_{pq} = a_{qp}$ $(p, q = 1, 2, \cdots, n)$ *a set of constants such that the symmetrical determinant*

$$\Delta_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

is not zero, $f_1(s), f_2(s), \cdots, f_n(s)$ *a set of integrable functions such that*

$$\int_0^1 [f_p(s)]^2 ds$$

is convergent. Then if

$$h_n(s, t) = \frac{1}{\Delta_n} \begin{vmatrix} k(s, t) & f_1(s) & f_2(s) & \cdots & f_n(s) \\ f_1(t) & a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f_n(t) & a_{n1} & \cdots & \cdots & a_{nn} \end{vmatrix} = \frac{F_n}{\Delta_n}$$

and if $h_{n-1}(s, t)$ *is derived from* $h_n(s, t)$ *by omitting the last row and column in each of the determinants* F_n *and* Δ_n, *the roots of the symmetric kernel* $h_{n-1}(s, t)$ *will be separated by those of* $h_n(s, t)$.

Let $D(\lambda)$, $D_{n-1}(\lambda)$, $D_n(\lambda)$ be the determinants of $k(s, t)$, $h_{n-1}(s, t)$, $h_n(s, t)$ respectively; then by a known formula*

$$D_n(\lambda) = \frac{D(\lambda)}{\Delta_n} \nabla_n(\lambda),$$

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* *Messenger of Mathematics,* 1908, p. 179.

where

$$\nabla_n(\lambda) = \begin{vmatrix} a_{11} + \lambda\tau_{11} & \cdots & a_{1n} + \lambda\tau_{1n} \\ \cdot & \cdots & \cdot \\ a_{n1} + \lambda\tau_{n1} & \cdots & a_{nn} + \lambda\tau_{nn} \end{vmatrix},$$

$$\tau_{pq} = \tau_{qp} = \int_0^1 f_p(s)\phi_q(s)ds = \int_0^1 \phi_p(s)f_q(s)ds,$$

and $\phi_q(s)$ is the solution of the equation

$$f_q(s) = \phi_q(s) - \lambda\int_0^1 k(s, t)\phi_q(t)dt.$$

Now let A_{pq} denote the cofactor of the constituent $a_{pq} + \lambda\tau_{pq}$ in the determinant $\nabla_n(\lambda)$; then by a property of determinants

$$A_{nn}A_{n-1\cdot n-1} - A^2_{n-1\cdot n} = \nabla_{n-2}(\lambda)\nabla_n(\lambda),$$

where $\nabla_{n-2}(\lambda)$ is derived from $\nabla_n(\lambda)$ by omitting the last two rows and columns.

Now $A_{nn} = \nabla_{n-1}(\lambda)$, hence when $\nabla_{n-1}(\lambda)$ vanishes $\nabla_{n-2}(\lambda)$ and $\nabla_n(\lambda)$ have opposite signs. The functions

$$\nabla_1(\lambda), \quad \nabla_2(\lambda), \quad \cdots, \quad \nabla_n(\lambda)$$

therefore form a Sturmian series and it will be seen presently that the roots of $\nabla_n(\lambda)$ separate those of $\nabla_{n-1}(\lambda)$, the roots of $\nabla_{n-1}(\lambda)$ separate those of $\nabla_{n-2}(\lambda)$, and so on.

Now the roots of $\nabla_n(\lambda) = 0$ are the same as those of $D_n(\lambda) = 0$ and I have shown in a former paper* that the roots of $D_1(\lambda)$ separate those of $D(\lambda)$, hence the functions

$$D(\lambda), \quad D_1(\lambda), \quad D_2(\lambda), \quad \cdots, \quad D_n(\lambda)$$

form a sequence such that the roots of any function in the sequence separate the roots of the preceding function.

If $\overset{+}{\lambda}_1, \overset{+}{\lambda}_2, \cdots, \overset{+}{\lambda}_{n+1}$ are the positive roots of $D(\lambda)$ arranged in order of magnitude, there will be n roots of $D_1(\lambda)$ arranged singly between the gaps, $n - 1$ roots of $D_2(\lambda)$ arranged between the gaps in this second set, and so on. It is clear then that there is at least one root of $D_n(\lambda)$ between $\overset{+}{\lambda}_1$ and $\overset{+}{\lambda}_{n+1}$; this includes Weyl's theorem. We have supposed that the roots of $D(\lambda)$ are all distinct, but the necessary modification for the case of multiple roots is easily introduced.

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[*] *Cambr. Phil. Trans.*, vol. 20 (1908), p. 374. The theorem is used again on p. 182.

If the constants a_{pq} are chosen so that the determinants Δ_n are all positive, $D_{n-2}(\lambda)$ and $D_n(\lambda)$ will have opposite signs when $D_{n-1}(\lambda)$ vanishes, and so the functions

$$D(\lambda),\quad D_1(\lambda),\quad D_2(\lambda),\quad \cdots,\quad D_n(\lambda)$$

will form a Sturmian sequence.

It has been stated that the roots of the functions $\nabla_n(\lambda)$ in the Sturmian sequence separate one another. This is not always true for a Sturmian sequence when the functions are not polynomials, but it can be shown to be true in the present case, as follows. Let $g_n(s)$, $g_n(t)$ be the cofactors of the constituents $f_n(t)$, $f_n(s)$ in the determinant F_n; then from the properties of determinants

$$F_{n-1}\cdot\Delta_n - g_n(s)g_n(t) = F_n\cdot\Delta_{n-1}.$$

Dividing out by $\Delta_{n-1}\Delta_n$, we have

$$h_n(s, t) = h_{n-1}(s, t) - \frac{g_n(s)g_n(t)}{\Delta_{n-1}\Delta_n}.$$

We can now apply the theorem mentioned before to this equation and deduce that the roots of $h_{n-1}(s, t)$ are separated by those of $h_n(s, t)$, there being one root of $h_n(s, t)$ between each consecutive pair of roots of $h_{n-1}(s, t)$.

BRYN MAWR COLLEGE,
November, 1911.

ON THE CUBES OF DETERMINANTS OF THE SECOND, THIRD, AND HIGHER ORDERS.

BY PROFESSOR ROBERT E. MORITZ.

(Read before the San Francisco Section of the American Mathematical Society, April 8, 1911.)

WHILE the square of a determinant of any order may be readily expressed as a determinant of the same order, I am not aware of the existence of a correspondingly simple method by means of which the cube of any determinant may be expressed in determinant form. For a determinant of the fourth order, Δ_4, we have indeed from a well-known property of determinants

$$\Delta_4^3 \equiv \Delta_4',$$

where Δ_4' is the determinant whose constituents are the cofactors of the corresponding constituents of Δ_4. Another isolated case may be derived from the following general theorem due to Faà di Bruno:

The determinant which has for its first row the constituents

$$x^n,\quad x^{n-1}y,\quad x^{n-2}y^2,\quad \cdots,\quad xy^{n-1},\quad y^n$$

and whose other rows are obtained by operating on the constituents of the first row successively by the symbolic operators

$$x'\frac{d}{dx}+y'\frac{d}{dy},\quad \frac{1}{1.2}\left(x'\frac{d}{dx}+y'\frac{d}{dy}\right)^{(2)},\quad \cdots,$$

$$\frac{1}{n!}\left(x'\frac{d}{dx}+y'\frac{d}{dy}\right)^{(n)}$$

is a power, namely the $\frac{1}{2}n(n+1)$th power, of the determinant $xy'-x'y$. For $n=2$, this theorem yields a determinant expression for $(xy'-x'y)^3$.

I

On multiplying each constituent of the determinant of the second order

$$\Delta_2 \equiv \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

by Δ_2, we obtain

$$\Delta_2^3 \equiv \Delta_2(a_1b_2-a_2b_1)^2 = \begin{vmatrix} a_1(a_1b_2-a_2b_1) & a_2(a_1b_2-a_2b_1) \\ b_1(a_1b_2-a_2b_1) & b_2(a_1b_2-a_2b_1) \end{vmatrix},$$

and this in turn is equivalent to the bordered determinant

$$\Delta_2^3 \equiv \frac{1}{a_1b_1}\begin{vmatrix} a_1b_1 & a_1b_2-a_2b_1 & a_2b_2 \\ 0 & a_1(a_1b_2-a_2b_1) & a_2(a_1b_2-a_2b_1) \\ 0 & b_1(a_1b_2-a_2b_1) & b_2(a_1b_2-a_2b_1) \end{vmatrix}.$$

Now subtract a_1 times the first row from the second, and add b_1 times the first row to the third, then

$$\Delta_2^3 \equiv \frac{1}{a_1b_1}\begin{vmatrix} a_1b_1 & a_1b_2-a_2b_1 & a_2b_2 \\ -a_1^2b_1 & -2a_1a_2b_1 & -a_2^2b_1 \\ a_1b_1^2 & 2a_1b_1b_2 & a_1b_2^2 \end{vmatrix},$$

or, after interchanging the first and second rows and removing factors,

$$(1)\qquad \Delta_2^3 \equiv \begin{vmatrix} a_1^2 & 2a_1a_2 & a_2^2 \\ a_1b_1 & a_1b_2 + a_2b_1 & a_2b_2 \\ b_1^2 & 2b_1b_2 & b_2^2 \end{vmatrix}.$$

By adding the first and third columns to the second and putting

$$a_3 = a_1 + a_2, \quad b_3 = b_1 + b_2,$$

we may write (1) in the form

$$(2)\qquad \Delta_2^3 \equiv - \begin{vmatrix} a_1^2 & a_2^2 & a_3^2 \\ a_1b_1 & a_2b_2 & a_3b_3 \\ b_1^2 & b_2^2 & b_3^2 \end{vmatrix}.$$

We have then the following

Rule for the cube of any determinant of the second order: Write down two lines of elements

$$\begin{array}{lll} a_1, & a_2, & a_3 (= a_1 + a_2), \\ b_1, & b_2, & b_3 (= b_1 + b_2), \end{array}$$

composed respectively of the constituents of the first row (or column) and their sum; the constituents of the second row (or column) and their sum, of any determinant of the second order. Its cube is the negative of the determinant of the third order the constituents of whose rows (or columns) are respectively the squares of the elements of the first line, the products of the elements of the first line by the corresponding elements of the second line, the squares of the elements of the second line.

II.

Next consider the determinant of the third order

$$\Delta_3 \equiv \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \equiv c_1c_2c_3 \begin{vmatrix} a_1' & a_2' & a_3' \\ b_1' & b_2' & b_3' \\ 1 & 1 & 1 \end{vmatrix},$$

where

$$a_1' = a_1/c_1, \quad a_2' = a_2/c_2, \quad a_3' = a_3/c_3, \quad b_1' = b_1/c_1, \text{ . etc.}$$

Expressed as a determinant of the second order,

$$\Delta_3 \equiv c_1c_2c_3 \begin{vmatrix} a_2' - a_3' & a_3' - a_1' \\ b_2' - b_3' & b_3' - b_1' \end{vmatrix},$$

from which by the use of (2)

$$\Delta_3^3 \equiv -(c_1c_2c_3)^3 \begin{vmatrix} x_1^2 & x_2^2 & x_3^2 \\ x_1y_1 & x_2y_2 & x_3y_3 \\ y_1^2 & y_2^2 & y_3^2 \end{vmatrix},$$

where

$$x_1 = a_2' - a_3', \quad x_2 = a_3' - a_1', \quad x_3 = a_1' - a_2',$$
$$y_1 = b_2' - b_3', \quad y_2 = b_3' - b_1', \quad y_3 = b_1' - b_2'.$$

But

$$x_1 = a_2' - a_3' = \frac{a_2}{c_2} - \frac{a_3}{c_3} = \frac{a_2c_3 - a_3c_2}{c_2c_3} = \frac{-B_1}{c_2c_3},$$

and similarly

$$x_2 = \frac{-B_2}{c_3c_1}, \quad x_3 = \frac{-B_3}{c_1c_2},$$

$$y_1 = \frac{A_1}{c_2c_3}, \quad y_2 = \frac{A_2}{c_3c_1}, \quad y_3 = \frac{A_3}{c_1c_2},$$

where the A's and B's are the cofactors of the corresponding constituents of Δ_3. Substituting these values, we obtain after a slight reduction

$$(3) \qquad \Delta_3^3 \equiv \frac{-1}{c_1c_2c_3} \begin{vmatrix} A_1^2 & A_2^2 & A_3^2 \\ A_1B_1 & A_2B_2 & A_3B_3 \\ B_1^2 & B_2^2 & B_3^2 \end{vmatrix}.$$

A case of special interest is that in which the c's are unity.

Thus

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 1 & 1 & 1 \end{vmatrix}^3$$

$$\equiv - \begin{vmatrix} (a_2-a_3)^2 & (a_3-a_1)^2 & (a_1-a_2)^2 \\ (a_2-a_3)(b_2-b_3) & (a_3-a_1)(b_3-b_1) & (a_1-a_2)(b_1-b_2) \\ (b_2-b_3)^2 & (b_3-b_1)^2 & (b_1-b_2)^2 \end{vmatrix}.$$

III.

Before considering the cubes of determinants of orders higher than the third, we will establish the following

THEOREM I. *Let* $\Delta_n \equiv (a_1 b_2 \cdots n_n)$ *represent any determinant of the nth order, and let S denote the substitution which replaces*

$$\left.\begin{aligned} &a_i \text{ by } (a_i k_{n+1}) = a_i k_{n+1} - a_{n+1} k_i, \\ &b_i \text{ by } (b_i k_{n+1}) = b_i k_{n+1} - b_{n+1} k_i, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &n_i \text{ by } (n_i k_{n+1}) = n_i k_{n+1} - n_{n+1} k_i \end{aligned}\right. \qquad (i = 1\ 2, \cdots, n);$$

then

$$S\Delta_n = k_{n+1}^{n-1} \Delta_{n+1} = k_{n+1}^{n-1} (a_1 b_2 \cdots n_n k_{n+1}),$$

where Δ_{n+1} *is the determinant of the* $(n+1)$*th order, formed by bordering* Δ_n *on the right by*

$$a_{n+1}, \quad b_{n+1}, \quad \cdots, \quad n_{n+1},$$

and on the bottom by

$$k_1, \quad k_2, \quad \cdots \quad k_n, \quad k_{n+1}.$$

Proof.—Making the substitution indicated, we have to show that

$$S\Delta_n \equiv \begin{vmatrix} (a_1 k_{n+1} - a_{n+1} k_1) & (a_2 k_{n+1} - a_{n+1} k_2) & \cdots & (a_n k_{n+1} - a_{n+1} k_n) \\ (b_1 k_{n+1} - b_{n+1} k_1) & (b_2 k_{n+1} - b_{n+1} k_2) & \cdots & (b_n k_{n+1} - b_{n+1} k_n) \\ \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \\ (n_1 k_{n+1} - n_{n+1} k_1) & (n_2 k_{n+1} - n_{n+1} k_2) & \cdots & (n_n k_{n+1} - n_{n+1} k_n) \end{vmatrix}$$

$$\equiv k_{n+1}^{n-1}\begin{vmatrix} a_1 & a_2 & \cdots & a_n & a_{n+1} \\ b_1 & b_2 & \cdots & b_n & b_{n+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ n_1 & n_2 & \cdots & n_n & n_{n+1} \\ k_1 & k_2 & \cdots & k_n & k_{n+1} \end{vmatrix}.$$

But this is merely the well-known identity which results from the reduction of any determinant of the $(n+1)$th order to one of order n.

Definition.—Let us call the substitution S which transforms any determinant $\Delta_n \equiv (a_1 b_2 \cdots n_n)$ into $\Delta_{n+1} = (a_1 b_2 \cdots n_n k_{n+1})$ the *substitution belonging to* Δ_n.

THEOREM II. *If any cofactor $H_i^{(n)}$ belonging to Δ_n is transformed by the substitution belonging to Δ_n, the result of the operation is the corresponding cofactor $H_i^{(n+1)}$ belonging to Δ_{n+1} multiplied by k_{n+1}^{n-2}.*

This theorem is an immediate consequence of Theorem I, since the cofactor $H_i^{(n)}$ of Δ_n, when bordered on the right by a_{n+1}, b_{n+1}, etc., and at the bottom by k_1, k_2, etc., becomes the cofactor $H_i^{(n+1)}$ of Δ_{n+1}. Similar conclusions may be arrived at with regard to the transformation of the minors of any order.

Theorems I and II, though they appear here as immediate consequences of well-known results, enable us to derive from certain given forms and relations other forms and relations. For any identity between a determinant and its minors will, when operated on by the substitution belonging to the given determinant, give rise to another identity between a determinant of the next higher order and its minors. These theorems, therefore, permit us to proceed from relations holding for determinants of the second or third orders to corresponding relations between determinants of any order.

IV.

We now apply the theorems of the preceding section to each member of equation (3). Δ_3 being of the third order, the substitution belonging to Δ_3 is

$$S = \left\{ \begin{matrix} a_i, & b_i, & c_i \\ (a_i d_4), & (b_i d_4), & (c_i d_4) \end{matrix} \right\} \quad (i = 1, 2, 3),$$

consequently

$$S\Delta_3 = d_4^2\Delta_4,$$

where

$$\Delta_4 = (a_1b_2c_3d_4),$$

$$SA_i = d_4A_i^{(4)},\quad SB_i = d_4B_i^{(4)},\quad SC_i = d_4C_i^{(4)}\qquad (i = 1,\ 2,\ 3)$$

where $A_i^{(4)}, B_i^{(4)}, C_i^{(4)}$, are the cofactors of the corresponding small letters in Δ_4. Finally $Sc_i = (c_id_4)$. Hence

$$S\Delta_3^3 = d_4^6\Delta_4^3 = \frac{-d_4^6}{(c_1d_4)(c_2d_4)(c_3d_4)}\begin{vmatrix} A_1^{(4)2} & A_2^{(4)2} & A_3^{(4)2} \\ A_1^{(4)}B_1^{(4)} & A_2^{(4)}B_2^{(4)} & A_3^{(4)}B_3^{(4)} \\ B_1^{(4)2} & B_2^{(4)2} & B_3^{(4)2} \end{vmatrix}$$

which, on removing the common factor d_4^6 and dropping the indices, becomes

$$(4)\qquad \Delta_4^3 \equiv \frac{-1}{(c_1d_4)(c_2d_4)(c_3d_4)}\begin{vmatrix} A_1^2 & A_2^2 & A_3^2 \\ A_1B_1 & A_2B_2 & A_3B_3 \\ B_1^2 & B_2^2 & B_3^2 \end{vmatrix}.$$

V.

The general case now offers no further difficulty. The substitution belonging to Δ_4 is

$$S = \begin{Bmatrix} a_i, & b_i, & c_i, & d_i \\ (a_ie_5), & (b_ie_5), & (c_ie_5), & (d_ie_5) \end{Bmatrix}\qquad (i = 1,\ 2,\ 3).$$

Operating on both members of (4) with S we obtain

$$(5)\qquad \Delta_5^3 \equiv \frac{-1}{(c_1d_4d_5)(c_2d_4e_5)(c_3d_4e_5)}\begin{vmatrix} A_1^2 & A_2^2 & A_3^2 \\ A_1B_1 & A_2B_2 & A_3B_3 \\ B_1^2 & B_2^2 & B_3^2 \end{vmatrix},$$

and hence by induction

$$(6)\qquad \Delta_n^3 \equiv \frac{-1}{(c_1d_4\cdots n_n)(c_2d_4\cdots n_n)(c_3d_4\cdots n_n)}\begin{vmatrix} A_1^2 & A_2^2 & A_3^2 \\ A_1B_1 & A_2B_2 & A_3B_3 \\ B_1^2 & B_2^2 & B_3^2 \end{vmatrix},$$

where the large letters are the cofactors of the corresponding small letters in Δ_n. It will be noticed that each member of (6) is of degree $3n$, as it should be.

UNIVERSITY OF WASHINGTON,
March, 1911.

NOTE ON THE MAXIMAL CYCLIC SUBGROUPS OF A GROUP OF ORDER p^m.

BY PROFESSOR G. A. MILLER.

(Read before the San Francisco Section of the American Mathematical Society, October 28, 1911.)

IF H is any non-invariant subgroup of a group G of order p^m, p being any prime number, it is well known that H is transformed into itself by at least one of its conjugates under G and hence by operators which are not contained in H.* If H is cyclic and not contained in a larger cyclic subgroup of G, it is said to be a *maximal cyclic subgroup of G*. In what follows we shall establish the

THEOREM: *A necessary and sufficient condition that every maximal cyclic subgroup of order* p^α *in a group* G *of order* p^m, $m > 3$, *is transformed into itself by no more than* $p^{\alpha+1}$ *operators of* G *is that* G *contains one and only one cyclic subgroup of order* p^{m-1}.

If we combine with this theorem some well-known properties of the groups of order p^m which contain operators of order p^{m-1}, it results that there are only three non-cyclic groups of order p^m which have the property that each of their maximal cyclic subgroups of order p^α is transformed into itself by only $p^{\alpha+1}$ operators of the group. These three groups are the three non-cyclic groups of order 2^m which involve one and only one cyclic subgroup of order 2^{m-1}.

To prove the theorem in question, we shall assume that G does not involve any operator of order p^{m-1}, since the groups of order p^m which contain operators of order p^{m-1} are so well known. We shall also assume in what follows that G satisfies the condition that each one of its maximal cyclic subgroups of order p^α is transformed into itself by exactly $p^{\alpha+1}$ operators of G, p^α being the order of any one of the maximal cyclic subgroup of G.

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* Cf. *American Journal of Mathematics*, vol. 23 (1901), p. 173.

If H is a maximal cyclic subgroup of order p^α contained in G, it must have $p^m/p^{\alpha+1}$ conjugates under G and these conjugates must involve

$$\frac{p^m}{p^{\alpha+1}}(p^\alpha - p^{\alpha-1}) = p^m\left(\frac{1}{p} - \frac{1}{p^2}\right)$$

distinct operators of order p^α. Since any complete set of conjugates of a non-cyclic group of order p^m is contained in a subgroup of order p^{m-1}, it results that these $p^m\left(\frac{1}{p} - \frac{1}{p^2}\right)$ operators of order p^α in G must generate a subgroup of order p^{m-1}. This subgroup includes a subgroup K of order p^{m-2} which does not involve any generator of the cyclic subgroups which are conjugate with H, but it is composed of all the other operators of the given subgroup of order p^{m-1}. Since G involves $p^m - p^{m-1}$ operators which are generators of maximal cyclic subgroups but are not found in the given subgroup of order p^m, it results that G *contains exactly* $p+1$ *distinct conjugate sets of maximal cyclic subgroups.*

As each of these distinct sets generates a subgroup of order p^{m-1} which involves K, it results that K is a characteristic subgroup of G which gives rise to an abelian quotient group of order p^2 and of type (1, 1). As all the operators of K are powers of larger operators which are not in K, it results that K cannot be cyclic. In fact, if K were cyclic, G would involve operators of order p^{m-1}, which is contrary to the assumption made above.

As K is non-cyclic it must involve more than one subgroup of order p^{m-3}, and the operators which are common to all of its subgroups of this order must constitute a characteristic subgroup with respect to which its quotient group is abelian and of type $(1, 1, 1, \cdots)$. Hence it results that G must have an invariant subgroup of index p^4 which gives rise to a quotient group such that the operators of K correspond to one of its non-cyclic subgroups of order p^2.

The operators of K which correspond to operators of order p in this quotient group cannot be powers of operators of higher order in K and hence they must all be pth powers of operators of G which are not in K. That is, this quotient group of order p^4 must be such that each of its operators of order p corresponding to K is a power of an operator of order p^2 in the rest of this quotient group. This implies that this quotient group is abelian and of type (2, 2) when $p = 2$, and when $p > 2$ it must contain at least p invariant cyclic subgroups of order p^2. As this is contrary to the fact that G contains $p+1$ conjugate sets which involve generating operators of its maximal cyclic subgroups, we have proved that we arrive at an absurdity by assuming that G does not involve any operator of order p^{m-1}, $m > 3$.

When $p > 2$ there are only two non-cyclic groups of order p^m which involve operators of order $p^{m-1}, m > 3$, and each of these clearly contains maximal cyclic subgroups of order p^α which are transformed into themselves by more than $p^{\alpha+1}$ operators of G. Hence it results that the three non-cyclic groups of order 2^m which were considered above in the second paragraph are the only non-cyclic groups of order p^m in which every maximal cyclic subgroup is transformed into itself by at most p times as many operators of the group as there are operators in this maximal subgroup. This completes the proof of the theorem in question, and hence we can assume that *every non-cyclic group of order p^m, with the exception of the three of order 2^m which involve one and only one cyclic subgroup of order 2^{m-1}, contains at least one maximal cyclic subgroup of order p^α which is transformed into itself by more than $p^{\alpha+1}$ operators of the group.*

UNIVERSITY OF ILLINOIS.

AN EXPRESSION FOR THE GENERAL TERM OF A RECURRING SERIES.

BY PROFESSOR TSURUICHI HAYASHI.

PROFESSOR Arthur Ranum has given in the BULLETIN, volume 17, No. 9, June, 1911, pages 457–461, an explicit form of the general term of a recurring series rationally in terms of the first few terms and the constants of the scale of relation. I will give here another more explicit and more convenient form without demonstration.

Let $u_0 + u_1 + u_2 + \dots + u_n + \dots$ be any recurring series of order n, and let

$$u_m = a_1 u_{m-1} + a_2 u_{m-2} + \cdots + a_n u_{m-n} \qquad (m \geqq n)$$

be its scale of relation. Then for any value of m not less than n,

$$u_m = \sum_{\nu=0}^{\nu=n-1} u_\nu A_\nu,$$

where

$$A_\nu = \sum a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n} \cdot \frac{(\alpha_1+\alpha_2+\cdots+\alpha_n-1)!}{\alpha_1!\,\alpha_2!\cdots\alpha_n!} \cdot (\alpha_n+\alpha_{n-1}+\cdots+\alpha_{n-\nu}),$$

the summation extending over all the positive integral and zero values of $\alpha_1, \alpha_2, \cdots, \alpha_n$, for which

$$\alpha_1 + 2\alpha_2 + 3\alpha_3 + \cdots + n\alpha_n = m - \nu.$$

Since the formula is true when $m = n$, and assuming it to be true for the values $n + l, n + l - 1, \cdots, n+1, n$ of m, we can prove it by mathematical induction to be true for the next value $n + l + 1$, without much difficulty, unless $l < n$. Even for the case $l<n$, a slight attention leads us to the same result.

More generally, it may also be shown that for any value of m greater than λ

$$u_m = \sum_{\nu=\lambda-n+1}^{\nu=\lambda} u_\nu A_\nu,$$

where

$$A_\nu = \sum a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n} \cdot \frac{(\alpha_1+\alpha_2+\cdots+\alpha_n-1)!}{\alpha_1!\,\alpha_2!\cdots\alpha_n!} \cdot (\alpha_n+\alpha_{n-1}+\cdots+\alpha_{\lambda-\nu+1}),$$

the summation extending over all the positive integral and zero values of $\alpha_1, \alpha_2, \cdots, \alpha_n$ for which

$$\alpha_1 + 2\alpha_2 + 3\alpha_3 + \cdots + n\alpha_n = m - \nu.$$

TÔHOKU IMPERIAL UNIVERSITY,
SENDAI, JAPAN,
July, 19 1.

SHORTER NOTICES.

Émile Picard. Biographie, Bibliographie analytique des Écrits. By ERNEST LEBON. Paris, Gauthier-Villars, 1910. viii+80 pp.

Paul Appell. Biographie, Bibliographie analytique des Écrits. By ERNEST LEBON. Paris, Gauthier-Villars, 1910. viii+71 pp.

THESE two new volumes of the Savants du Jour are entirely similar in plan and arrangement to their two predecessors, which were devoted to Henri Poincaré and Gaston Darboux respectively. They contain, besides the brief biographical sketches of Picard and Appell by M. Lebon, a complete classified list of their publications. The titles and references to the place of publication are in many cases followed by a brief synopsis of the content of the corresponding paper. The following notes taken from these volumes may be of some interest to readers of the BULLETIN.

Charles Émile Picard was born in Paris on July 24, 1856. He entered the École Normale in 1874 and received his doctorate in 1877. During his three years at the École Normale he was inspired by Darboux; under the guidance of the latter he produced his first memoir (presented to the Academy of Sciences, January, 1877) and his doctor's dissertation. Nearly all of his subsequent publications, however, relate closely to pure analysis; even his famous researches on algebraic surfaces are properly classified as pure analysis. In 1879, at the early age of 23 years, he published the two important theorems on integral functions that bear his name. These investigations attracted the attention of Hermite and led to a long and intimate friendship which resulted in 1881 in the marriage of Picard and the daughter of Hermite. After a short experience in teaching at Toulouse he was recalled to Paris in 1881, where he has remained ever since. He was elected to the Academy of Sciences in 1889, and was its president in 1910. He was awarded the Prix Poncelet in 1886 and the Grand Prix des Sciences mathématiques in 1888 for his work on the theory of algebraic functions of two variables. His published work numbers 341 titles.

Paul Émile Appell was born in Strasbourg, September 27,

1855. He entered the École Normale in 1873 and obtained his doctorate in 1876. His first published work relates to projective geometry and shows the influence of his teacher, Chasles. His dissertation falls into this period. But beginning with 1877 he, like Picard, devoted himself almost exclusively to the field of analysis. After a brief experience in the provinces, he was recalled to Paris in 1881, where he has remained. He was elected to the Academy of Sciences in 1892. The list of his publications contains 306 titles.

J. W. YOUNG.

Vermeintliche Beweise des Fermatschen Satzes. Besprochen von A. FLECK, PH. MAENNSCHEN, O. PERRON. *Archiv der Mathematik und Physik*, Bände 14–18.

Über das letzte Fermatsche Theorem. Von BENNO LIND. *Abhandlungen zur Geschichte der mathematischen Wissenschaften*, Heft XXVI$_2$, pp. 21–65, 1910.

THE age of circle squarers had hardly come to an end (one still meets sporadic cases here and there) when a new period, the age of the "Fermatists," arose. This genus suddenly received a tremendous boom through the "Wolfskehlsche Preisstiftung" by which a prize of 100,000 Marks is offered to him who first proves the great theorem of Fermat. And in its wake there have come a host who do not have the least ambition to add anything to mathematical knowledge but merely lust after the prize money. Their number will doubtlessly reach many thousands within the 99 years for which the prize is established.

The editors of the *Archiv* have opened their pages to a discussion of the proofs that might be submitted; and it is the pleasant (?) duty of Messrs. Fleck, Maennschen, and Perron to peruse these and point out to each writer the errors which lurk in his work. Of the 111 attempts at a proof that are discussed to date in the *Archiv*, none is successful and not one adds anything which may be of value in bringing the solution of the problem nearer completion. Fermat stated in a marginal note that *the equation*

$$x^n + y^n = z^n \tag{1}$$

has no solution in integers, for all values of n which are greater than 2, and that he possessed a most wonderful proof of this theorem; but his proof, if such existed, has never been discovered. Each of these latter-day Fermatists believes that he has found a like remarkable proof, and is ready to grasp the prize, only to have his hopes shattered by an exposure of his errors. Some of them, nothing daunted, have made a second and even a third attempt, only to fail each time. What, besides the prize, has driven these engineers, teachers, captains of artillery, building inspectors, doctors, book-keepers, preachers, apothecaries, judges, merchants, cabinet ministers, and representatives of many other vocations and of many nationalities, to waste time, paper, and ink on the worthless stuff they have turned out? The reviewer almost believes it is because they consider the problem as a sort of challenge to their intelligence — it is stated so clearly and in such few words — it sounds so familiar — just a little extension of the Pythagorean theorem — and it asks for a solution in integers, the kind of numbers we played with in our childhood days. Carried away by these impulses, they forget that such mathematicians as Abel, Cauchy, Euler, Legendre, Kummer, and a score of others have attempted the problem before them; they do not consider that they lack the subtlety of mind, the "Scharfsinn" which the subject requires; they do not even think it worth while to find out what progress has been made along this line by previous writers, but each one starts afresh and attempts, by mighty path-breaking methods, to arrive at the goal within the compass of 50 pages or of 2 pages or even on a post-card.

And wherein do these Fermatists fail — what is the nature of their errors? Disregarding completely such puerile algebraic blunders as are usually attributed to our college freshmen, almost all the errors may be classified under one of the following types: 1) Assumption of the converse of a theorem where only the direct theorem has been proved; 2) omission of steps in the proof of a theorem whose proof in turn is either impossible or as difficult as the original theorem; 3) assumption that because certain expressions satisfy an equation, these are necessarily the only ones that do so; 4) reasoning in a circle; 5) drawing conclusions for general values from theorems which have only been proven for a particular value; 6) confusion of algebraic and numerical factoring; 7) ignorance of simple factor theorems or divisibility of numbers; 8) errors in elementary theory of congruences; 9) assumption of particular forms of the unknown quantities; 10) assumption without proof that a certain complicated expression of fractional form which has been set up must be an integer. These types of errors are about equally numerous — the first five are errors in logic, the last five are more closely related to the theory of numbers. May the *Archiv* continue its good work in the exposure of these pseudo-proofs. Let us hope that the editors will not grow tired, within the next 96 years, of their self-imposed task, which does have its compensations on the humorous side if on no other.

Lind, in his monograph, does not attempt to prove Fermat's theorem. It is his aim to give a complete algebraic survey in simplified form of everything that has been done by elementary methods towards a solution of the problem, to point out the directions which might lead to further developments, and to give a historical review of the problem since the time of Fermat. Though not complete, still his collection of the material from so many varied sources and its presentation as a whole is very noteworthy. His work is divided into four parts. Part 1 gives the algebraic analysis of the problem by elementary methods. It contains a host of theorems, with their proof, which have been deduced concerning the relations that must exist between x, y, z, and n if the equation (1) is to hold. We first have the well-known Abel equations expressing x, y, z and the three very important functions of the form

$$\frac{x^n + y^n}{x + y}, \quad \frac{z^n - y^n}{z - y}, \quad \frac{z^n - x^n}{z - x}$$

in terms of the nth powers of 6 quantities $a, b, c, \alpha, \beta, \gamma$. These relations are the starting point of all the later investigations.

Lind includes some original work in this part and in part 4, and here he comes to grief. Fleck discusses this portion of the monograph in the *Archiv,* volume 16, 1st Heft (1910), and in reply to Lind's objections to some of Fleck's criticisms, the latter goes into more detail in volume 18, 1st Heft (1911). And so we have another exhibit in the *Archiv's* gallery of exposures. Several of the theorems which Lind gives as his own in part 1 are false in themselves or their proofs are false. In part 4, he attempts to make a most important addition to the development of the problem, viz., that the equation (1) has no solution in integers x, y, z prime to n, and that in case

$n = 6m + 1$ one of these must be divisible by $3n^2$. Unfortunately, he bases his proof upon two of his theorems dealing only with the most elementary theory of congruences (in particular, if $x \equiv a \pmod{3}$, then $x^n \equiv a^n \pmod{3^2}$ for all odd primes n), which Fleck points out are both false, and thus Lind's original work is mostly worthless.

Part 2 contains a bibliography of the subject, citing 183 articles. Part 3 is the historical review. The first attempt to treat Fermat's theorem in general is due to Claude Jaquemets (1651–1729), but he did not advance very far. The very important Abel equations were first set up by Barlow (1811), and then independently by Abel, Kummer, Legendre, Lindemann, F. Lucas, and Stäckel. These men also established a large number of other formulas and theorems. One of the most important theorems that Abel stated without proof was, that if equation (1) is to have integral solutions, then none of the quantities x, y, z, $x + y$, $z - y$, etc., can be a prime. In regard to this theorem, Talbot, in 1857, proved that the two larger of the unknowns cannot be primes. It was not until 1905 that Sauer showed that the smallest of the unknowns cannot be a prime. We should add here that Cauchy, Lebesgue, Liouville, and Gruenert have also arrived at very many interesting results.

As far as the proof of the impossibility of equation (1) in integers for special values of n is concerned, Euler in 1738 gave a proof for $n = 4$, and 22 years later for $n = 3$. The case $n = 5$ is an interesting one, for in 1825 Dirichlet showed that if equation (1) has solutions, then one of the unknowns must be divisible by 5, and that this unknown cannot be odd; Legendre followed this by showing (by the same method) that this unknown cannot be even, and thus completed the case for $n = 5$. In 1840 Lamé proved the case $n = 7$. Many others have given proofs covering these cases. In 1823 Legendre showed that if (1) has solutions, then for all $n < 100$ one of the unknowns must be divisible by n; but it remained for Dickson (*Quarterly Journal*, volume 40 (1908)) to prove that (1) has no integral solutions *all prime to n* for every $n < 6{,}857$, and for the larger primes $< 7{,}000$. But the year 1847 brought the richest harvest for the problem, for then Kummer, with the help of the theory of prime ideals proved that (1) has no integral solutions for *all values of n* which are odd primes and which are contained as factors in the numerators of the first $\frac{1}{2}(n - 3)$

Bernoulli numbers — this includes all odd primes < 100, with the exception of 37, 59, 67. Ten years later he extended this to another series of exponents including these three exceptional numbers, so that the Fermat theorem was proven for all values of $n > 2$ and $\leqq 100$. Kummer received the Paris prize of 1850 for his beautiful work.

In 1909, an award of 1000 Marks was made from the Wolfskehl foundation to G. Wieferich for his proof that Fermat's equation has no solutions all prime to n unless $2^{n-1} \equiv 1 \bmod n^2$. And here the problem rests.

JOSEPH LIPKE.

Solid Geometry. By H. E. SLAUGHT and N. J. LENNES. Allyn and Bacon, Boston, 1911. vi+190 pp.

THIS book follows the plane geometry of the textbook series of the authors. It is divided into seven chapters entitled lines and planes in space, prisms and cylinders, pyramids and cones, regular and similar polyhedrons, the sphere, variable geometric magnitudes, and theory of limits.

The logical phase of the development of solid geometry, as here treated, is a great improvement over that usually found in our textbooks. Many of the more fundamental principles are formally stated as axioms. The first striking example of this is Axiom III: "If two planes have a point in common, then they have at least another point in common." This fundamental theorem of three-dimensional geometry has usually been kept as obscure as possible. In all, ten axioms are thus stated.

A brief treatment of sines, cosines, and tangents, and a few theorems on the projection of lines and of areas are introduced. Some of the theorems on trihedral angles are deferred to the chapter on the sphere, where they are related to the theory of spherical triangles. Euler's theorem is stated without proof, the usual faulty proof being inserted as an exercise in which the error in the proof is to be shown. The definition of polar spherical triangles is made completely. The proof of the theorem that the shortest path between two points on a sphere is the arc of a great circle joining the points is made to depend on the concept of the length of a curve on a sphere as the limit of the sum of the lengths of small arcs of great circles — a somewhat different notion from the limit of the sum of lengths of the chords, which has been previously used in the book for the length of a curve. In the chapter on variable geometric magnitudes, graphs are used to show the relation between volume, surface, and edge of a regular tetrahedron, cube, etc. The final chapter on limits is excellent, and introduces many new ideas not usually presented.

The usual faulty proof of the theorem on the volume of an oblique prism is retained. The minor errors are not numerous, but the following have been noticed: On page 51, line 16, the word cylinder is used where prism is meant. On page 85, line 8 from the bottom, d is used in place of $\frac{1}{2}sd$.

The book is very teachable, and taken altogether is a marked improvement over the usual text.

F. W. OWENS.

First Course in Algebra. By H. E. HAWKES, W. A. LUBY, and F. C. TOUTON. Ginn and Company. vii+334 pp.

THE purpose of the authors as stated in the preface, namely, "to build up a text book thoroughly modern, scientifically exact, teachable and suited to the needs and to the ability of the boy and girl of fourteen," has been in a large measure accomplished. The topics and problems, with a few exceptions which will be noted later, seem to have been chosen with excellent judgment. The idea of reasoning with symbols instead of numbers is introduced gradually but insistently; transposition is explained by means of addition and subtraction and the student is taught the actual use of equations before the term equation is defined at all. The lists of examples in factoring, in linear and quadratic equations, and several other topics, are sufficiently varied and extensive to give the student a thorough drill in elementary mathematical reasoning and manipulation. The authors have made good their intention as stated in the preface to use clear and exact English throughout the book. Typographical errors are few, but we have noted the following: on page 139, line 5, read fraction instead of fractions; replace 3 by $\sqrt{3}$ in the answer to example 1, page 321. We would suggest the use of the word may instead of should in line 7, page 211. The sentence beginning in line 6, page 262, is spoiled somewhat by the presence of the two words graph and figure. It would seem better to use a capital G in the last word on page 263, and similarly in example 18, page 264. One might question the value of asking in example 21, page 322, for a proof that the product of conjugate imaginary numbers is real, after conjugates have been defined on the preceding page as imaginaries whose product is real. Well-founded objection might be made to examples 9, 10, and 11 on page 212. They involve a principle of physics with which the first-year algebra student will quite certainly be unfamiliar. The force of this objection depends of course on the degree to which we wish to encourage the student to accept principles or facts on a merely plausible explanation, which is probably all that would ever be given in class. The authors have in a number of cases introduced principles quite beyond the province of the book to discuss and have then built problems on these principles. While this is a matter on which there may be room for argument, it is the reviewer's opinion that there is no such dearth of desirable material for problems for the first year's work in algebra that one need bring in totally strange ideas from physics and geometry in order to have plenty. The subject of graphs is emphasized to a degree that makes certain parts of the book resemble an intuitional analytic geometry. The proof that a straight line may be defined by an equation of the first degree is not too difficult for the student after a very brief study of plane geometry. Until he is prepared for some such proof would it not be wiser to let the graph alone? We feel that when the student is beginning a subject that is essentially logical, all unnecessary appeals to his intuition should be avoided.

J. V. McKelvey.

CORRECTIONS.

On page 66 of the current volume of the Bulletin the writer gave the following theorem:

If on the interval ab

$$\sum_{n=0}^{\infty} U_n(x) = f(x),$$

$U_i(x)$ $(i = 0, \cdots, \infty)$ and $f(x)$ being continuous on the interval ab, then in order that $\sum_{n=0}^{\infty} U_n(x)$ shall be uniformly convergent on the interval ab it is necessary and sufficient that for any x_i on ab, any arbitrary number δ (however small), and any arbitrary integer N there is an integer $N'(i, \delta, N)$ greater than N, which satisfies the following condition:

On the interval $x_i - \delta$, $x_i + \delta$ there is a value x_i' of x such that for every $N_i'' > N'(i, \delta, N)$

$$\left|\sum_{n=0}^{N'(i,\,\delta,\,N)} U_n(x_i') - f(x_i')\right| \geqq \left|\sum_{n=0}^{N_i''} U_n(x) - f(x)\right|.$$

for every value of x on the interval $x_i - \delta$, $x_i + \delta$ which also lies on ab.

In its present form this theorem is incorrect so far as it relates to a sufficient condition. The error consists in making N_i depend on δ. Instead we must require that an N_i exists such that for any δ subsequently chosen the conditions given shall hold. In giving the proof of the sufficiency of the condition this is the form in which the hypothesis was used.

However the condition as modified is not necessary. Hence we have failed to obtain a necessary and sufficient condition. The theorem as stated in its original form gives a necessary condition, while the amended form gives a sufficient condition. Professor Osgood very kindly called the writer's attention to this error.

N. J. LENNES.

IN my paper on "Invariant conditions that a p-ary form may have multiple linear factors" in the BULLETIN, volume 17, No. 9 (June, 1911) the condition $R_3 = 0$ in equations (7), page 455, should be replaced by $R_3' = 0$, where R_3' is the resultant of $x_2^2\phi_{1x_1/x_2}$ and the greatest common divisor (linear) of $x_2^3\phi_{0x_1/x_2}$ and $x_2^2\phi_0'{}_{x_1/x_2}$.

O. E. GLENN.

THE following typographical corrections should be made in my review of Holton's Shop Mathematics, in the December number of the BULLETIN:

Page 138, footnote. For Warnier read Warner.

Page 140. In the formulas $P_1 = \pi/P$, $P = 16$, $N = DP \cos Y$, for P read p.

C. N. HASKINS.

NOTES.

At the annual meeting of the London mathematical society held on November 9 the following papers were read: By J. E. Campbell, "The invariants of the linear partial differential equation of the second order, having two independent variables"; by H. Hilton, "On invariants of a canonical substitution"; by C. T. Bennett, "The system of lines on a cubic surface"; by G. H. Hardy and J. E. Littlewood, "The relations between Borel's and Cesaro's methods of summation"; by W. P. Milne, "A method of establishing the 27-line configuration on a cubic surface"; by H. Bateman, "Mathematical analogues of mental phenomena."

At the meeting of the Edinburgh Mathematical Society on November 10, the following papers were read: By W. P. Milne, "The system of cubic curves circumscribing two triangles and apolar to them"; by W. P. Milne, "An easy geometrical representation of the sextic covariant of a binary quartic"; by T. Hugh Miller, "On the numerical calculation of natural logarithms."

At the meeting of December 8 the following papers were read: By William Peddie: "The mechanical solution of an n-ic equation, and exhibition of a model for the determination of the roots of a cubic; by D. M. Y. Sommerville: "Einstein's principle of relativity in its kinematical aspect" and "Note on Legendre's and Bertrand's proofs of the parallel postulate by infinite areas."

The officers of the society for the year 1911–1912 are: president, D. M. Y. Sommerville; vice-president, A. G. Burgess; secretary, Peter Cowrie.

The first congress of Russian professors of mathematics will be held at St. Petersburg January 9–16, 1912.

A new journal, entitled the "*Vector,*" has been established at Warsaw. It will be devoted to mathematical and physical science in general and especially to questions of method and pedagogy.

The publishing house of Gauthier-Villars in Paris announces the following mathematical works in press: Works of Hermite, volume 3; Works of Laplace, volume 14; Stoffaes, Course in higher mathematics, volume 2, curves and surfaces and differential equations. Five further volumes in the Borel series of monographs on the theory of functions, volume 3 of Goursat's Cour d'analyse and volume 4 of Picard's Traité are nearly ready for the press.

THE seventh edition of the catalogue of mathematical models has just appeared from the firm of Martin Schilling, Leipzig. It is arranged in two parts, the first giving descriptions of the several series of models and a scientific explanation of each, the second containing an illustrated price catalogue arranged according to subject matter. The catalogue occupies 172 pages, and contains descriptions of 377 models in stock.

The two latest additions are an apparatus for generating a hyperboloid of revolution by turning one line about another skew to it, by Professor K. DOEHLEMANN, of the University of Munich, and a plaster model of a surface of constant latitude, by Dr. E. MEISSNER of the technical school at Zürich.

THE publication of an Introduction to the Lie theory of one-parameter groups, with applications to the solution of differential equations, by Dr. A. COHEN, of John Hopkins University, is announced by D. C. Heath and Company.

THE Jean Reynaud prize of ten thousand francs, awarded by the Paris academy of sciences every five years, has been bestowed this year on Professor EMILE PICARD, for his contributions to mathematics.

AT the annual meeting of the Royal society of London held in November a royal medal was awarded to Professor GEORGE CHRYSTAL, of the University of Edinburgh, for his researches in mathematics and physics, and a Copley medal was awarded to Professor Sir G. H. DARWIN, of the University of Cambridge, for his contributions to the theory of astronomical evolution.

PROFESSOR E. W. HOBSON, of the University of Cambridge, has been elected a member of the academy of sciences of Halle.

AT the University of Liège M. J. DERUYTS has been appointed professor of analysis and M. E. FAIRON professor of geometry, algebra, and methodology.

DR. TRAYNARD, of the University of Lille, has been appointed professor of differential and integral calculus at the University of Besançon.

Dr. Maillet has been appointed professor of mathematics and mechanics at the Ecole des ponts et chaussées at Paris.

The College entrance examination board has appointed two groups of examiners in mathematics for 1912. The first will prepare the questions in algebra, and is composed of H. E. Hawkes, chairman, C. R. MacInnes, and W. A. Francis; the second, composed of V. Snyder, chairman, P. F. Smith, and J. T. Rorer, is to prepare the questions in geometry and trigonometry.

At the University of Washington, Mr. G. I. Gavett has been promoted to an assistant professorship of mathematics. Miss A. D. Biddle and Mr. John Whitmore have been appointed instructors in mathematics.

Mr. J. S. Lockner, of Lehigh University, has been appointed instructor in mathematics at the Case school of applied science at Cleveland, Ohio.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Armbrecht (A.). Zum grossen Fermatschen Satz! 2te, verbesserte Auflage. Dresden, Kohler, 1911. 8vo. 4 pp. M. 0.50

Bonola (R.). Non-Euclidean geometry. A critical and historical study of its development. Authorized English translation, with additional appendices, by H. S. Carslaw. Chicago, Open Court Pub. Co., 1911. 10+258 pp.

Borghino (G. N.). Metodo generale di estrazione delle radici e di soluzione delle equazioni, con numerose tavole per l'estrazione delle radici e la soluzione delle equazioni fino al 10° grado. Introduzione alla logica matematica. Torino, Paravia, 1911. 8vo. 192 pp. L. 2.50

Botto (A.). Soluzione geometrica del problema relativo alla duplicazione del cubo Torino, Paravia, 1911. 8vo. 13 pp

Bromwich (T. J. I'a). Elementary integrals. London, Macmillan, 1911. 8vo. Sewed. 1s.

Burali-Forti (C.). Sopra una formola generale per la trasformazione di integrali di omografie vettoriali. (Nota.) Torino, Bona, 1911. 8vo. 23 pp.

Calegari (A.). Brevi nozioni di calcolo infinitesimale. Livorno, Giusti, 1911. 16mo. 139 pp. L. 2.00

Carslaw (H. S.) See Bonola (R.).

Encyclopédie des sciences mathématiques pures et appliquées. Edition française. Tome I, volume 4: Statistique, assurances, économie politique. Fascicule 4. Leipzig, Teubner, 1911. 8vo. pp. 481–640. M. 6.00

Ender (A.) Die konformen Raumtransformationen. 1ter Teil. (Progr.) Waidhofen, 1911.

Euleri (L.). Opera omnia. Sub auspiciis societatis scientarium naturalium helveticæ edenda curaverunt F. Rudio, A. Krazer, P. Stäckel. Series III: Opera physica, miscellanea, epistolæ. Vol. III, Pars 1: Dioptrica. Ed. Emil Cherbuliez. Vol. 1: Continens librum primum, libri secundi sectionem primam et secundam. Leipzig, Teubner, 1911. 8vo. 510 pp. M. 24.00

Frankl (M. W.). Der Verhältniskalkül. Ein Beitrag zur logischen Algorithmik und zur Gegenstandstheorie. (Progr.) 1911. 8vo. 23 pp.

Fuchs (C. A.). Die Aehnlichkeitsbeziehungen zweier und mehrerer Kreise. (Progr.) Komotan, 1911. 8vo. 23 pp.

Gidály (R.). Die Hauptmethoden der Konstruktion einer Fläche zweiter Ordnung aus neun gegebenen Punkten. (Progr.) Wiener-Neustadt, 1911. 8vo. 40 pp.

Gläser (K.). Ebene Katakausten. (Progr.) Linz, 1911. 8vo. 7 pp.

Grünberger (E.). Das Schnittwinkelproblem dreier Kreise. (Progr.) Plan, 1911. 8vo. 9 pp.

Grunert (W.). Beiträge zur Theorie der Bewegungen und Umlegungen im Raum. (Diss.) Königsberg, 1911. 8vo. 46 pp.

Hämmerle (H.). Die Stirlingsche Formel. (Progr.) Krems, 1911. 8vo. 20 pp.

Haupt (O.). Untersuchungen über Oszillations-theoreme. (Diss.) Würzburg, 1911. 8vo. 50 pp.

Heckel (J.). Über trigonometrische Reihen. (Progr.) Reichenberg, 1911. 8vo. 31 pp.

Henderson (A.). The twenty-seven lines upon the cubic surface. London, Cambridge University Press, 1911. 8vo. 108 pp. 4s. 6d.

Herz (N.). Philosophische Konzeption und mathematische Analyse in der Weltbetrachtung. Bologna, Zanichelli, 1911. 8vo. 32 pp.

Holz (R.). Über die Bestimmung der Schnittpunkte von Geraden und Kegelschnitten. (Zyklographische Grundgedanken.) (Progr.) Teplitz-Schönau, 1911. 8vo. 8 pp.

Jankcutz (L.). Über die in zwei Kegelschnitte zerfallende Durchdringungskurve zweier Flächen zweiten Grades. (Progr.) Klagenfurt, 1911. 8vo. 16 pp.

Lindborg (G.). Elementerna af derivatkalkylen jämte problemsamling. Uppsala, Appelberg, 1910. 8vo. 4+68 pp. Kr. 2.00

Lussan (E.). Essai de démonstration générale du théorème de Fermat. Paris, Gauthier-Villars, 1911.

Melichar (P. J.). Určeni geom. místa pólu řezu svazku plochy 2 st. vzhledem ku svazku paprskovému. (Progr.) Kroměřiži, 1911. 8vo. 6 pp.

Miceli (L.). Condizioni di divisibilità di un numero N per un numero a, scritti nel sistema di numerazione indo-arabica. Matera, Conti, 1911. 8vo. 8 pp.

Millar (W. J.). Inledning till differential- och integralkalkylen med exempel på dess anvendning inom mekaniken. Översat från 8e engelska upplaga av E. Johnson. Stockholm, Norstedt, 1911. 8vo. 71 pp. Kr. 1.90

Milne (J. J.). An elementary treatise on cross ratio geometry, with historical notes. London, Cambridge University Press, 1911. 8vo. 312 pp. 6s.

Mühlendyck (O.). Klassifikation der regelmässig symmetrischen Flächen fünfter Ordnung. (Diss.) Göttingen, 1911. 8vo. 60 pp.

Ondraczek (H.). Der Dirichlet'sche Satz über die linearen Funktionen im Zahlenkörper der Determinante "+5." (Progr.) Wien, 1911. 8vo. 45 pp.

Persson (K.). Die invarianten Gebilde erster Ordnung bei projektiven Transformationen der Ebene und des Raumes mit Anwendung auf die Klassifikation der eingliedrigen, projektiven Gruppen der Ebene und des Raumes. (Diss.) Heidelberg, 1911. 8vo. 45 pp.

Pezzo (P. Del). Principi di geometria proiettiva: lezioni dettate nell' università di Napoli nell' anno 1910–1911. Napoli, Alvano, 1911. 8vo. 360 pp.

Poincaré (H.). Calcul des probabilités. Rédaction de A. Quiquet. 2e édition. Paris, Gauthier-Villars, 1912. 8vo. 340 pp. Fr. 12.00

Rajakowitsch (J.). Elemente der Funktionenlehre. (Fortsetzung.) Geschrieben für Schüler. (Progr.) Triest, 1911. 8vo. 13 pp.

Ranucci (D. N.). Risoluzione dell' equazione $x^n - Ay^n = \pm 1$, con una nuova dimostrazione dell' ultimo teorema di Fermat. Roma, Pontificia, 1911. 8vo. 22 pp. L. 2.00

Rulf (F.). Behandlung des Plückerschen Konoides auf Grund einer neuen Definition. (Progr.) Wien, 1911. 8vo. 18 pp.

Schmidt (J.). Der Infinitesimalkalkül. (Progr.) Wien, 1911. 8vo. 28 pp.

Schwacha (P. B.). Über die Existenz und Anzahl der Wurzeln der Kongruenz $\sum_{i=0}^{n} c_i x^i \equiv 0 \pmod{m}$. (Progr.) Wilhering, 1911. 8vo. 30 pp.

Stahl (H.). Abriss einer Theorie der algebraischen Funktionen einer Veränderlichen in neuer Fassung. Nachgelassene Schrift in Verbindung mit E. Löffler herausgegeben von M. Noether. Leipzig, Teubner, 1911. 8vo. 4+103 pp. M. 5.00

STUDY (E.). Vorlesungen über ausgewählte Gegenstände der Geometrie. Erstes Heft: Ebene analytische Kurven und zu ihnen gehörige Abbildungen. Leipzig, Teubner, 1911. 8vo. 4+126 pp. M. 4.80

TIERSCH (W.). Ein elementarer Beweis des Fermatschen Satzes. Buttstädt i. Th., 1911.

TOISOUL (J.). Particularités utiles et intéressantes sur les nombres et leurs combinaisons. Namur, Lambert-DeRoisin, 1911. 16mo. 103 pp. Fr. 1.20

II. ELEMENTARY MATHEMATICS.

ANDOYER (H.). Nouvelles tables trigonométriques fondamentales contenant les logarithmes des lignes trigonométriques de centième en centième du quadrant avec dix-sept décimales, de neuf en neuf minutes avec quinze décimales, et de dix en dix secondes avec quatorze décimales. Paris, Hermann, 1911. 4to. 32 + 604 pp. Fr. 30.00

BATES (E. L.) and CHARLESWORTH (F.). Practical mathematics and geometry. Part III: Advanced Course. London, Batsford, 1911. 8vo. 338 pp. 3s.

BRUMBY (A. G.). Key to elementary practical geometry. (Normal Tutorial series.) London, Normal Press, 1911. 8vo. Sewed. 1s.

CAJORI (F.). Notes on the history of geometry and algebra. Boston, Heath, 1911. 8vo. 25 pp.

CASARI (F.). Aritmetica, geometria, computisteria: esercizi, problemi e nozioni, per il corso popolare delle scuole elementari, classe sesta. Milano, Signorelli, 1911. 16mo. 187 pp. L. 1.00.

CHAMPION (H. H.) and LANE (J. A. C.). A school geometry. London, Rivingtons, 1911. 8vo. 296 pp. 3s. 6d.

CHARLESWORTH (F.). See BATES (E. L.).

CHENARD (H.). Géométrie. Pour les écoles pratiques d'industrie et sections industrielles (1re année). Paris, Delagrave, 1910. 18mo. 303 pp.

COMBEROUSSE (C. DE). Cours de mathématiques à l'usage des candidats à l'Ecole polytechnique, etc. 5e édition, revue et augmentée. Tome 1er, 2e partie; algèbre élémentaire. Paris, Gauthier-Villars, 1911. 8vo. 295 pp. Fr. 6.00

COMMISSAIRE (H.). Leçons d'algèbre et de trigonométrie conformes aux programmes du 27 juillet 1906. (Classes de mathématiques A et B.) Paris, Masson, 1911. 8vo. 6+590+9 pp. Fr. 7.00

CRANTZ (P.). Arithmetik und Algebra zum Selbstunterricht. 2ter Teil. 2te Auflage. (Aus Natur und Geisteswelt. Sammlung wissenschaftlich-gemeinverständliche Darstellungen. Nr. 205.) Leipzig, Teubner, 1911. 8vo. 4+124 pp. Cloth. M. 1.25

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Denis-Guibert (H.). Etude sur les cinquante pas géométriques dans nos colonies. Mayenne, Colin, 1911. 8vo. 140 pp.

Dietsch (C.). See Sievert (H.).

Eckhardt (E.). Éléments de trigonométrie rectiligne. Hoogstraeten, Van Hoof-Roelaus, 1904. 8vo. 35 pp. Fr. 0.50

Focke und Kratz (M.). Lehrbuch der ebenen Trigonometrie nebst den Grundzugen der spharischen Trigonometrie zum Gebrauche an Gymnasien, Realgymnasien, Oberrealschulen, und anderen hoheren Lehranstalten. 12te, verbesserte Auflage, besorgt von I. Linneborn. Munster, Universitäts-Buchhandlung, 1911. 8vo. 4+90 pp. M. 1.80

Fossati (L.). Piccola guida pratica di aritmetica, geometria e computisteria, ad uso delle classi 3a, 4a, 5a e 6a elementari. Palermo, Sandron, 1911. 16mo. 206 pp. L. 1.25

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Geipel (G.) und Hecht (C.). Mathematisches Lehrbuch für hohere weibliche Bildungsanstalten. Lehrbuch der Mathematik und Aufgabensammlung. Bearbeitet von G. Geipel. Bielefeld, Velhagen und Klasing, 1911. 8vo. 4+137 pp. Cloth. M. 1.80

Genau (A.) und Krömeke (J.). Geometrie für das Lyzeum und die gymnasialen Kurse der Mädchenbildungsanstalten. Im Anschluss an die Raumlehre fur Lehrerinnenbildungsanstalten und höhere Mädchenschulen von A. Genau. Leipzig, Reisland, 1911. 8vo. 8+167 pp. Cloth. M. 2.00

Girndt (M.). Raumlehre, I. (Der Unterricht an Baugewerkschulen, 20.) 4te, neu bearbeitete Auflage. Leipzig, Teubner, 1911. M. 1.80

Grévy (A.). Géométrie plane, conforme au programme du 27 juillet 1905 à l'usage des élèves du 1er cycle *A* (classes de quatrième *A* et troisième *A*.) Paris, Vuibert, 1912. 16mo. 8+283 pp.

Hagström (K. L.). Matematiska uppgifter i studentexamen för latingymnasiet med anvisningar och svar. Linköping, Carlson, 1911. 8vo. 16 pp. Kr. 0.25

Halsted (G. B.). Géométrie rationelle. Traité élémentaire de la science de l'espace. Traduction française par P. Barbarin. Paris, Gauthier-Villars, 1911. 8vo. 3+301 pp. Fr. 6.50

Hecht (C.). See Geipel (G.).

Hedström (J. S.) och Rendahl (C.). Trigonometri för läroverken. Stockholm, Bonnier, 1911. 8vo. 6+121 pp. Kr. 2.50

Kommerell (B.) und Kommerell (K.). Analytische Geometrie. Für den Schulgebrauch bearbeitet. 1ter Teil. Tubingen, Laupp, 1911. 8vo. 8+192 pp. M. 2.40

Kratz (M.). See Focke.

Krömecke (J.). See Genau (A.).

Lärobok i rakning och geometri för armén. II: Algebra och geometri. Stockholm, Norstedt, 1910. 8vo. 62+61 pp. Kr. 1.25

Laue (J. A. C.). See Champion (H. H.).

Lesser (O.). See Schwab (K.).

Link (T.). See Vollkommer (M.).

Manuel d'algèbre et de trigonométrie, par une réunion de professeurs. Paris, Poussielgue, 1911. 18mo. 8+216 pp.

Martello (D.). Esercizi di aritmetica e geometria, per la quarta classe elementare. Milano, Nugoli 1911. 16mo. 160 pp. L. 0.80

Meiser (W.). Losungen zu Aufgaben aus der algebraischen Analysis, nach J. Lieblein bearbeitet. (Fortsetzung.) Nurnberg, Korn. 1911. 8vo. 3+59 pp. M. 2.00

Mönkemeyer (K.) und Rüsewald (K.). Lehr- und Übungsbuch der Mathematik fur hohere Madchenschulen. Auf Grund der neuen Lehrpläne bearbeitet. 4tes Heft. Klasse 1. Leipzig, Quelle und Meyer, 1911. 8vo. 4+80 pp. M. 1.00

Morawetz (J.). Vierstellige logarithmische und trigonometrische Tafeln, nebst einigen Hilfstafeln. Zum Schulgebrauch herausgegeben. Leipzig, Freytag, 1911. 8vo. 52 pp. M. 0.80

Otto (F.), Petri (W.) and Ziegler (J.). Mathematik fur Lyzeen. 2ter Teil. Leipzig, Hirt und Sohn, 1911. 8vo. 6+167 pp. Cloth. M. 2.25

Persiani (O.). Elementi di geometria, compilati ad uso della quarta ginnasiale. Terza edizione, con notevoli aggiunte e modificazioni. Roma, Pontificia, 1911. 16mo. 102 pp. L. 0.80

Petri (W.). See Otto (F.).

Porta (F.). I complementi di matematica per l'ammissione alla reale accademia militare e alla reale accademia navale. 5a edizione. 8vo. 295 pp. L. 5.00

Rendahl (C.). See Hedström (J. S.).

Runquist (N. F.). Repetitionskurs i analytisk geometri. Malmo, Envall, 1911. 8vo. 4+86 pp. Kr. 1.50

Schmehl (C.). Lehrbuch der ebenen Trigonometrie fur hohere Lehranstalten. Giessen, Roth, 1911. 8vo. 6+152 pp. M. 2.50

Schneider (J.). Geometrie leicht gemacht! Ein Lehr- und Übungsbuch der Geometrie. Bamberg, Buchner, 1911. 8vo. 113 pp. M. 1.80

Schwab (K.) and Lesser (O.). Mathematisches Unterrichtswerk zum Gebrauche an den hoheren Lehranstalten für die weibliche Jugend. Bearbeitet von M. Linnich. Leipzig, Freytag, 1911. 8vo. 228 pp. Cloth. M. 3.00

Sievert (H.) und Dietsch (C.). Lehrbuch der Elementar-Geometrie zum Gebrauche an Mittelschulen und beim Selbstunterrichte. 1ter Teil. 1te Abteilung: Geometrie der Ebene. 5te Auflage. Leipzig, Diechert, 1912. 8vo. 6+183 pp. M. 2.40

—— Lehrbuch der Elementar-Geometrie zum Gebrauche an Mittelschulen und beim Selbstunterrichte. 1ter Teil. 2te Abteilung: Geometrie des Raumes. 3te Auflage. Leipzig, Diechert, 1912. 8vo. 6+183 pp. M. 2.40

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THE EIGHTEENTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

The eighteenth annual meeting of the Society was held in New York City on Wednesday and Thursday, December 27–28, 1911, the programme occupying two sessions on each day. The total attendance numbered about seventy-five, including the following sixty-four members:

Dr. C. S. Atchison, Mr. H. Bateman, Professor W. J. Berry, Professor G. D. Birkhoff, Professor Joseph Bowden, Professor B. H. Camp, Professor W. M. Carruth, Dr. A. S. Chessin, Professor C. W. Cobb, Dr. Emily Coddington, Professor F. N. Cole, Professor J. L. Coolidge, Dr. L. L. Dines, Mr. E. P. R. Duval, Professor L. P. Eisenhart, Professor T. C. Esty, Professor F. C. Ferry, Professor J. C. Fields, President H. B. Fine, Professor W. B. Fite, Mr. Meyer Gaba, Professor A. S. Gale, Professor O. E. Glenn, Mr. G. H. Graves, Professor C. C. Grove, Professor J. G. Hardy, Professor C. N. Haskins, Professor H. E. Hawkes, Professor L. A. Howland, Professor L. S. Hulburt, Dr. Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Dr. D. D. Leib, Dr. N. J. Lennes, Mr. Joseph Lipke, Professor W. R. Longley, Professor C. R. MacInnes, Professor J. H. Maclagan-Wedderburn, Professor James Maclay, Professor G. A. Miller, Professor C. L. E. Moore, Professor Richard Morris, Professor A. D. Pitcher, Professor Arthur Ranum, Dr. H. W. Reddick, Professor R. G. D. Richardson, Dr. J. E. Rowe, Mr. L. P. Siceloff, Mr. C. G. Simpson, Professor P. F. Smith, Mr. W. M. Smith, Professor H. W. Tyler, Professor Anna L. Van Benschoten, Professor Oswald Veblen, Mr. H. E. Webb, Professor H. S. White, Mr. E. E. Whitford, Dr. W. A. Wilson, Professor E. B. Wilson, Professor F. S. Woods, Professor J. W. Young.

President Fine occupied the chair, being relieved at the Wednesday afternoon session by Ex-President White. The Council announced the election of the following new members of the Society: Professor Ida Barney, Rollins College; Professor Louis Brand, University of Cincinnati; Professor C. W. Cobb, Amherst College; Professor J. C. Fitterer, University of Wyoming; Mr. G. H. Graves, Columbia University; Dr.

Solomon Lefschetz, University of Nebraska; Mr. G. H. Light, Purdue University; Mr. E. S. Palmer, Rutgers College; Professor E. R. Smith, Pennsylvania State College. Eight applications for membership in the Society were received.

The annual dinner on Wednesday evening was a very pleasant occasion, forty-two members being present. These informal gatherings have long been recognized as one of the most attractive features of the meetings.

The reports of the Treasurer, Auditing Committee, and Librarian have appeared in the Annual Register. The membership of the Society has risen to 668, including 62 life members. The total attendance of members at meetings during the year just passed was 423; the number of papers read was 217. At the annual election 197 votes were cast. The Society's library now contains 3,871 volumes, beside some 500 unbound dissertations. The Treasurer's report shows a balance of $8,723.89, but the increase is offset by outstanding bills amounting to about $550. The income from sales of the Society's publications during the year was $1,513.66. The life membership fund now amounts to $4,137.17.

With the great growth of the Society's activities it will soon become necessary to make some provision for clerical services, the capacity of the several offices being already overstrained in this direction. The Society has outgrown the stage in which its officers could attend personally to all the details of administration.

At the annual election, which closed on Thursday morning, the following officers and other members of the Council were chosen:

Vice-Presidents,	Professor H. F. BLICHFELDT, Professor HENRY TABER.
Secretary,	Professor F. N. COLE.
Treasurer,	Professor J. H. TANNER.
Librarian,	Professor D. E. SMITH.

Committee of Publication,
Professor F. N. COLE,
Professor E. W. BROWN,
Professor VIRGIL SNYDER.

Members of the Council to serve until December, 1914.

Professor A. B. COBLE,
Professor E. W. DAVIS,
Professor OSWALD VEBLEN,
Professor E. B. WILSON.

The following papers were read at this meeting:

(1) Mr. W. M. SMITH: "A characterization of isogonal and equitangential trajectories."

(2) Professor C. L. E. MOORE: "Surfaces in hyperspace which have a tangent line with three-point contact passing through each point."

(3) Dr. J. E. ROWE: "How to find a set of invariants for any rational curve of odd order."

(4) Dr. J. E. ROWE: "A covariant point of the R^4, and a special canonical form."

(5) Dr. R. L. MOORE: "On sufficient conditions that an integral equation of the second kind shall have a continuous solution."

(6) Professor E. B. WILSON: "Some mathematical aspects of relativity."

(7) Professor EDWARD KASNER: "Families of surfaces related to an arbitrary deformation of space."

(8) Dr. H. B. PHILLIPS and Professor C. L. E. MOORE: "Algebra of plane projective geometry."

(9) Professor ANNA L. VAN BENSCHOTEN: "Products of quadric inversions and linear transformations in space" (preliminary report).

(10) Professor ARTHUR RANUM: "N-dimensional spreads generated by ∞^1 flats."

(11) Professor O. E. GLENN: "Generalizations of a theorem on reducible quantics, due to Eisenstein."

(12) Professor F. R. SHARPE: "Finite groups of birational transformations in the plane."

(13) Professor JOHN EIESLAND: "On a flat spread-sphere geometry in an odd dimensional n-space."

(14) Professor C. N. MOORE: "The summability of the double Fourier series, with applications."

(15) Professor S. E. SLOCUM: "A general formula for torsional deflection."

(16) Professor G. A. MILLER: "Groups which contain a given number of operators whose orders are powers of the same prime."

(17) Professor R. G. D. RICHARDSON: "Theorems of oscillation for three self-adjoint linear differential equations of the second order with three parameters."

(18) Professor L. A. HOWLAND: "Points of undulation of algebraic plane curves."

(19) Professor C. N. Haskins: "Note on certain selective integrals."

(20) Professor L. P. Eisenhart: "Ruled surfaces with isotropic generators."

(21) Professor J. W. Young: "On algebras defined by groups of transformations."

(22) Professor J. W. Young: "A generalization to 3-space and to n-space of the inversion geometry in a plane" (preliminary communication).

(23) Dr. L. L. Silverman: "On absolute or unconditional summability."

(24) Dr. L. L. Silverman: "Tests for Cesàro summability."

(25) Professor W. B. Fite: "Note on a collineation group in n variables."

(26) Professor L. P. Eisenhart: "Congruences of minimal lines in a four-space."

In the absence of the authors the papers of Dr. Moore, Dr. Sharpe, Professor Eiesland, Professor C. N. Moore, Professor Slocum, and Dr. Silverman were read by title. Abstracts of the papers follow below.

1. Cesàro has shown that in any complete isogonal system the locus of the centers of curvature of the single infinity of trajectories passing through the same point is a straight line. Similarly, Scheffers has proved that in any complete equitangential system those trajectories tangent to the same straight line have their centers of curvature upon a straight line. The former is here referred to as the Cesàro property and the latter as the Scheffers property.

In any doubly infinite system of plane curves any one of the single infinity of trajectories through a point of the plane has a certain curvature γ. Let N denote the rate of variation of γ along the normal, the direction of the consecutive trajectory remaining the same, and let γ_1, N_1 denote the corresponding quantities for that trajectory which is orthogonal to the first. Mr. Smith shows that a doubly infinite system of plane curves whose single infinity of trajectories through any point possess the property expressed by

$$N + N_1 = \gamma^2 + \gamma_1^2$$

together with the Cesàro property is a complete isogonal system.

If it possesses this property together with the Scheffers property, it is a complete equitangential system.

2. In this paper Professor Moore shows that the coordinates of a surface in hyperspace, having through each point a tangent line with three-point contact, must satisfy a partial differential equation of parabolic type. If the coordinates satisfy such an equation, the surface will be of the kind mentioned. Similar results are obtained for other spreads in hyperspace.

3. An algebraically complete system of invariants of the rational plane quartic has been found by Dr. Rowe in a previous paper presented to the Society September 6, 1910. (See *Transactions*, July, 1911.) This enables us to find a set of invariants for any rational plane curve of odd order, which is called R^{2n+1}. Every R^{2n+1} has a covariant rational line quartic ρ^4, whose parametric equations may be written down, and from these the four fundamental invariants for each ρ^4 may be calculated. This work has been carried out in detail for the R^5.

4. In Dr. Rowe's second paper a covariant point of the R^4, not on the R^4, is found which possesses the following properties: (1) tangents to R^4 from this point are apolar to the six flex parameters and to the q's each taken three times; (2) the line cutting out the q's on R^4 and the line cutting them out of Stahl's conic N meet in this point; (3) the flex lines of two cubic osculants whose parameters are given by a binary quadratic meet in this point. The coordinates of the point are of degree four in the coefficients of the parametric expressions of the R^4.

In the second part of the paper a special canonical form of the R^4 is discussed which enables us to express in terms of the four fundamental invariants conditions on the R^4 which are rather difficult to attack directly. For instance, the condition that two of a set of covariant parameters, cut out of the R^4 by a covariant curve, form a node is easily handled by means of this canonical form. The flecnode and tacnode conditions occur as special cases of this.

5. Making use of a theorem on which he reported at the April, 1911, meeting of the Society (cf. BULLETIN, volume 17, No. 10, page 513), Dr. Moore proposes to show that

if $f(x)$ is continuous throughout the interval

$$I: \quad a \leqq x \leqq b$$

and $k(x, y)$ is a limited function of x and y in the square

$$S: \quad a \leqq x \leqq b, \quad a \leqq y \leqq b,$$

and if the Fredholm determinant D is not 0, and finally if on each parallel to the axis of x or of y the points of discontinuity, if any, of $k(x, y)$ which lie in S form a point set of linear measure 0, then the integral equation of the second kind

$$u(x) = f(x) + \int_a^b k(x, y)u(y)dy$$

has a solution $u(x)$ which is continuous throughout I.

6. Professor Wilson comments on some points of a mathematical nature in a memoir which he and Professor G. N. Lewis, of the Massachusetts Institute of Technology, are writing on the principle of relativity. The occurrence of a parabolic non-euclidean geometry and of a corresponding non-circular vector analysis, and the possibility of the use of potentials in four dimensions are the two chief subjects treated. The former was in part set forth and in part implied in Minkowski's work; the latter has been suggested by Sommerfeld.

7. Professor Kasner discusses the existence and determination of surfaces which are affected conformally, or more specially isometrically, by a given deformation of space. Unless the transformation of space is conformal there cannot be more than six simply infinite families of surfaces affected conformally; and unless the deformation is a displacement there cannot be more than two families affected isometrically.

8. From the assumption that the sum of two points obeys the ordinary laws of algebraic addition, Dr. Phillips and Professor Moore show that there are just two kinds of addition possible, depending upon the interpretation given to a number times a point. If λA is coincident in position with A, then the sum of two points is the harmonic of a fixed line f with respect to the two points. If λA differs in position from A, then the

sum of two points becomes similar to the ordinary vector sum. This second kind of addition leads to a sort of vector analysis in which vector points appear as well as vector lines. Both additions have two fundamental or exceptional elements consisting of a line and a point.

Distance is defined as a scalar quantity uniquely determined by two points and such that distances *AB* along a line are proportional to the corresponding vectors *AB*. On these assumptions is built a theory of distance. As a consequence of the definition, to each line of the plane is assigned a definite positive direction. Angle is taken as the exact dual of distance. In the geometry in which distance and angle are defined as above, the locus of points equidistant from *A* is a line. Also it is found that any three parts of a triangle determine it.

It is assumed for the area of a triangle that, holding the vertex fixed and moving the base of constant length along a line, it will remain constant. It is then found that the area must be a multiple of the sum of the sides.

The product *AB* of two unit points is defined as the segment connecting the two points. The product of three unit points is defined as the area of the triangle having the three points as vertices. Similar definitions are given for the products of lines.

9. In this report Miss Van Benschoten presents some types of (2, 2), (2, 3), (3, 3), and (3, 4) birational transformations which are obtained as products of quadric inversions and linear transformations in space of three dimensions. It is proved that the (3, 3) transformation in which for each variable its reciprocal is substituted can be resolved in $\frac{1}{2}n(n+1)$ different ways into the product of $n-1$ inversions and a linear transformation in space of n dimensions.

10. This paper is a continuation of the one read by Professor Ranum at the summer meeting of the Society. Taking the ith tangent spread of the jth focal spread of the kth tangent spread of a given spread S, and letting i, j, k vary, we obtain a finite system of spreads determined by S. A number of properties of this system are found. Among the theorems proved is the following:

If S and its 1st, $\cdots$, $(i-1)$-th tangent spreads are all of rank r and its ith tangent spread S_1 of rank $s_1 < r$, while its

1st focal spread S_2 is of rank $s_2 < r$ ($s_1 + s_2 \geqq r$), and if either T_1, the $(i + 1)$-th focal spread of S_1, or T_2, the $(i + 1)$-th tangent spread of S_2, is of rank s_1+s_2-r, then the other is also of the same rank and T_1 is the $(i + 1)$-th focal spread of T_2. This remarkable case first occurs in 9-dimensional space.

11. Eisenstein's theorem may be stated as follows: A necessary condition that a binary form α_x^m with integral coefficients a_0, $a_1, \cdots, a_m$, all of whose coefficients except a_0 are divisible by a prime number q, should be reducible in the absolute field is that

$$(1) \qquad a_m \equiv 0 \pmod{q^2}.$$

Professor Glenn proves corresponding theorems for p-ary forms. He finds that the criteria imposed by these theorems become sharper as p increases, due to the fact that the number of conditions corresponding to (1) is, in the general case, an increasing function of p. A class of theorems on the existence of a factor of a p-ary form whose order lies within limits is established. A new method of resolving a reducible p-ary form is developed. It is proved for a linearly factorable p-ary form that, if a certain related binary form is linearly factorable in a given domain, the p-ary form is reducible in the same domain, though not necessarily completely (linearly) reducible unless the factors of the binary form are all simple. Necessary arithmetical conditions that the p-ary form be completely reducible in that domain are then developed. These take the form of higher order congruences in one unknown. The last section of the paper is devoted to the theory of these congruences.

12. Noether[*] has shown that a single plane can be depicted upon a double plane so that the double curve is a sextic with two coincident triple points. S. Kantor[†] has proved that the finite group of birational transformations having 8 fundamental points in the single plane is isomorphic with the group of linear transformations which leave the sextic in the double plane invariant. Wiman[‡] used the Grassmann depiction of the single plane upon a cubic surface and thence upon the double plane.

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[*] *Erlangen Berichte*, 1878.

[†] *Acta Math.*, vol. 19.

[‡] *Math. Annalen*, vol. 48.

He also completely enumerated the linear transformations of the sextic. In Professor Sharpe's paper a method is developed for determining the transformation of the single plane which corresponds to a given linear transformation of the double plane. It is known that there are 120 tritangent conics to the sextic which are also tangent to the sextic at the triple point. It is shown that each conic leads to a partial determination of the cubic surface and that the cubic surface can then be completely determined in 28 ways. In a special case the 120 conics and the transformations of the single plane are completely determined.

13. Professor Eiesland's paper is in abstract as follows: A generalization of Lie's sphere geometry becomes possible if, instead of using the straight line as space element, we introduce a flat spread (hyperplane) of $\frac{1}{2}(n-1)$ dimensions. In fact, since to spheres that touch must correspond intersecting flats, the condition for such intersection must be unique; moreover, $\frac{1}{2}(n-1)$ must be an integer, hence, a flat spread-sphere geometry is possible only in a space of an odd number of dimensions. It is also necessary that the number of essential parameters of the flat shall be equal to $n+1$, which is the number of parameters of the sphere.

It has been shown that for an odd space Lie's contact transformation can be generalized and that this new transformation converts the ∞^n spheres in S_{n-1} (n even)

$$\sum_1^{n-1}(x_i-\xi_i)^2+\xi_n^2=0$$

into the ∞^n flat spreads

$$x_i=ay_i+b_i,\quad z=\sum_1^{n-2}c_jy_j+d\quad (i=1,2,\cdots,\tfrac{1}{2}(n-2))$$

in a space

$$\bar{S}_{n-1}(x_1,y_1,x_2,y_2,\cdots,x_{\frac{1}{2}(n-2)},y_{\frac{1}{2}(n-2)})$$

with the following relation between the ξ_i's and the parameters a, b_i, c_i, d:

$$\begin{gathered}\xi_1=-\tfrac{1}{2}(b_1+c_1),\quad \xi_2=\tfrac{1}{2}i(b_1-c_1),\cdots,\\ \xi_{n-2}=\tfrac{1}{2}i(b_{\frac{1}{2}(n-2)}-c_{\frac{1}{2}(n-2)}),\\ \xi_{n-1}=\tfrac{1}{2}(a-d),\quad \xi_n=-\tfrac{1}{2}i(a+d).\end{gathered}\tag{1}$$

The condition that two flats shall intersect is

$$(\Sigma c_i' - c_i)(b_i' - b_i) + (d' - d)(a - a') = 0,$$

which, on introducing the parameters ξ_i from (1), becomes the condition that two spheres shall touch.

The transformation in question is

$$\begin{aligned}
X_1 &= \frac{(y_1 - p_1)(\Sigma y_i q_i - z) - (1 - \Sigma p_i y_i)(x_1 + q_1)}{2(1 - \Sigma p_i y_i)}, \\
X_2 &= i\,\frac{(y_1 + p_1)(z - \Sigma y_i q_i) + (1 - \Sigma p_i y_i)(x_1 - q_1)}{2(1 - \Sigma p_i y_i)}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
X_{n-1} &= \frac{\Sigma y_i q_i - z}{1 - \Sigma p_i y_i}, \\
P_1 &= \frac{y_1 - p_1}{1 + \Sigma p_i y_i}, \quad P_2 = \frac{-\,i(y_1 + p_1)}{1 + \Sigma p_i y_i}, \cdots, \\
P_{n-2} &= \frac{-\,i(y_{\frac{1}{2}(n-2)} + p_{\frac{1}{2}(n-2)})}{1 + \Sigma p_i y_i}.
\end{aligned} \tag{2}$$

Furthermore, this transformation converts the lines of curvature on an $n - 2$ dimensional spread in S_{n-1} into curves on the corresponding surface which are the analogues of asymptotic curves on a surface in 3-space. These lines have therefore been called asymptotic lines. Through any point on the surface pass $n - 2$ such lines and no more.

Finally *all* the contact transformations of this kind are considered. It is found that they are obtained by superposing the transformation (2) on the transformations of the group of contact transformations in S_{n-1} which change spheres into spheres, and also on the duality transformation. To the before mentioned group in S_{n-1} corresponds in $\bar{S}_{n-1}$ a group which leaves invariant the ∞^n flats, and transforms asymptotic lines into asymptotic line. This group is a subgroup of a group which is similar to the group of contact transformations discussed by Lie in the second volume of his large work on transformation group, chapter 25, where he shows that the transformations of the group leave invariant a hypersurface

$$z = a + \Sigma b_i x_i + \sum_{ik}^{1,\ldots,n-1} c_{ik} x_i x_k.$$

14. Fejér's idea of studying the summability of the Fourier development of an arbitrary function can be extended to the development in a double Fourier series of a function of two variables. In Professor Moore's paper this extension is carried out, and it is shown that the Fourier development of a function of two variables, $f(x, y)$, continuous in the region $(-\pi \leqq x \leqq \pi, -\pi \leqq y \leqq \pi)$, except for a finite number of lines of discontinuity consisting of parallels to the axes or of curves that cut such parallels in a finite number of points, will be summable to the value of the function at every point in the interior of the region at which the function is continuous, provided the discontinuities of the function are of such a nature that the double integrals in the coefficients of the Fourier's series converge. Along the lines of discontinuity the series will be summable at points where the function approaches a definite value as we approach the point from either side of the line, and its value will be half way between the limiting values provided the line of discontinuity has a tangent at the point. The definition of summability adopted is analogous to that adopted by Cesàro for ordinary series. The detailed formulas for positive integral orders of summability have been given in the abstract of a paper previously presented to the Society*; the present paper considers only summability of order one.

Furthermore, the present paper shows how the results stated above, together with the results of the previous paper just referred to, can be applied to the solution of certain problems in the flow of heat.

15. In a previous article (*Journal of the Franklin Institute*, April, 1911) Professor Slocum has deduced a general formula for shearing deflection, and has also given a new and simplified proof of the Fraenkel formula for flexural deflection. In the present paper this work on deflection is completed by the derivation of a general formula for torsional deflection.

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* BULLETIN, vol. 17 (1911), p. 530. There is an error in condition (*b*) of the theorem stated in this abstract. The two first equations of this condition should be replaced by the equations

$$\lim_{m=\infty} m^k \sum_{j=\mu_m}^{j=\nu_m} j^k \,|\, f_{m,j}(\alpha, \beta) \,| = 0, \quad \lim_{n=\infty} n^k \sum_{i=\mu_n}^{i=\nu_n} i^k \,|\, f_{i,n}(\alpha, \beta) \,| = 0,$$

where μ_m and ν_m vary with m, and μ_n and ν_n vary with n in any manner whatsoever.

Let M = torsional moment, or torque, acting in a plane perpendicular to the axis of the piece,
ϑ = torsional deflection, or angle of twist, produced by M,
q = shearing stress per unit area of cross section,
q_x, q_y = rectangular components of q,
φ = angular, or shearing, distortion corresponding to the stress q,
G = modulus of rigidity, or shear modulus,
W = internal work of deformation,
A = area of cross section,
l = length of member.

The internal work of deformation is first found in the form

$$W = \frac{1}{2G}\int_0^l\int_0^A q^2 dA dx.$$

For a prismatic piece of constant cross section this becomes

$$W = \frac{l}{2G}\int_0^A q^2 dA.$$

The following theorem is then proved:

If ϑ denotes the torsional deflection produced by a torque M, and W the total work of deformation arising from a torque $M + K$, then

$$\vartheta = \left[\frac{dW}{dK}\right]_{k=0}.$$

Inserting in this expression the value previously found for W in terms of the shearing stress q, the required formula for torsional deflection is obtained, namely

$$\vartheta = \frac{l}{GM}\int_0^A q^2 dA.$$

To illustrate the application of this formula, it is then used for calculating the torsional deflection of prismatic pieces the cross sections of which are respectively circular, elliptical, triangular, rectangular, and square.

16. If p^m is the highest power of the prime number p which divides the order of a group G, the number of the operators of G whose orders are powers of p is a multiple of p^m, according

to a theorem due to Frobenius. It is, however, not possible to construct a group in which this number is an arbitrary multiple of p^m. In fact, it is at least p^{m+1} whenever it exceeds p^m, and it is at least $2p^{m+1} - p^m$ whenever it exceeds p^{m+1}.

Professor Miller proves, as a special case of a general theorem, that a necessary and sufficient condition that any group contains exactly $p + 1$ Sylow subgroups of order p^m is that it contains exactly p^{m+1} operators whose orders are powers of p. He also established the following theorems: If every possible pair of Sylow subgroups of order p^m in a given group has p^{m-1} common operators, then all of these subgroups have the *same* p^{m-1} common operators. If any group contains less than $(p + 1)^2$ Sylow subgroups of order p^m, then all the operators which are common to some pair of these subgroups must be common to all of them. If G contains exactly $2p^{m+1}p^m$ operators whose orders are powers of p, it contains also exactly $2p + 1$ subgroups of order p^m and all of these involve the same p^{m-1} operators.

17. At the April meeting of the Society, Professor Richardson gave necessary conditions and sufficient conditions for the existence of solutions of the Klein oscillation problem for two equations with two parameters. In the present paper he discusses the problem for three equations with three parameters,

$$[p_i(x_i)u_i'(x_i)]' + q_i(x_i)u_i(x_i) + [\lambda A_{i1}(x_i) + \mu A_{i2}(x_i) + \nu A_{i3}(x_i)]u_i(x_i) = 0, \quad p_i(x_i) > 0 \quad (i = 1, 2, 3).$$

In order that there exist solutions $u_1(x_1)$, $u_2(x_2)$, $u_3(x_3)$ which satisfy the boundary conditions

$$u_1(a_1) = u_1(b_1) = 0, \quad u_2(a_2) = u_2(b_2) = 0,$$
$$u_3(a_3) = u_3(b_3) = 0,$$

and oscillate l, m, n times inspectively, it is necessary that parameter values $\lambda^{(1)}$, $\mu^{(1)}$, $\nu^{(1)}$; $\lambda^{(2)}$, $\mu^{(2)}$, $\nu^{(2)}$; $\lambda^{(3)}$, $\mu^{(3)}$, $\nu^{(3)}$ exist such that $\lambda^{(i)}A_{i1} + \mu^{(i)}A_{i2} + \nu^{(i)}A_{i3} \leqq 0$ for at least one value of the variable x_i in the interval a_ib_i and that $\lambda^{(i)}A_{j1} + \mu^{(i)}A_{j2} + \nu^{(i)}A_{j3} \geqq 0$, $\lambda^{(i)}A_{k1} + \mu^{(i)}A_{k2} + \nu^{(i)}A_{k3} = 0$ for all values of the variables x_j, x_k in the intervals a_jb_j, a_kb_k respectively $(i = 1, 2, 3;\ j = 1, 2, 3;\ k = 1, 2, 3;\ i \neq j \neq k)$. The sufficient conditions for $l \geqq l_1$, $n \geqq n_1$, $m \geqq m_1$ ($l_1 = 0$ when $q = 0$) are the same when the sign $\leqq$ is replaced by the sign $<$.

Another sufficient condition (in this case for all values of l, m, n) is that the determinant of the A's be different from zero for all values of the variables. The solution is then unique.

This paper is to appear in the *Mathematische Annalen.*

18. Points of undulation or, in general, points of an algebraic plane curve at which the tangent has contact of order higher than 2 receive little attention in elementary discussions of the theory of such curves. Or, in other words, although the behavior of the Hessian in multiple points of the curve is treated as fully as may be by elementary methods, the behavior of the curve in multiple points of the Hessian, the consequence of contact between curve and Hessian, or, more generally, of multiplicity of intersection of curve and Hessian greater than $3k(k-1)$ in an ordinary k-fold point are not treated at all.

The object of Professor Howland's paper is to show that these questions can be discussed with the same degree of completeness by elementary methods as can the theory of ordinary inflexions. Among other results the formula $\Sigma(\sigma - 1) = 3n(n-2)$, which applies to curves without multiple points, is extended to curves having ordinary multiple points, in the form $\Sigma(\sigma - 1) = 3n(n-1) - \Sigma 3k(k-1)$, where σ is the order of contact of curve and tangent at each point and Σ is extended to all points of the curve.

19. Landau[*] and Lebesgue[†] have recently discussed certain formulas by means of which a given continuous function can be expressed to a preassigned degree of approximation by means of an analytic function. The latter, in particular, has given an exceedingly general theorem by means of which such approximate representations can be constructed. The note of Professor Haskins discusses a similar theorem, somewhat less general than that of Lebesgue, which, however, throws some light on the mechanism of the process of representation of a function by means of a definite integral. The theorem is: If in the interval $a \leqq x \leqq b$ $f(x)$ is continuous,

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[*] *Rendiconti del Circolo Matematico di Palermo*, vol. 25 (1908), p. 337.

[†] Ibid., vol. 26 (1908), p. 325.

and $\varphi(x)$ is continuous and positive and attains its maximum value M only once in the interval, namely at the point $x=x_0$, then

$$\lim_{n=+\infty} \frac{\int_a^b f(x)\phi(x)^n dx}{\int_a^b \phi(x)^n dx} = f(x_0).$$

The theorem is closely related to a theorem of Stieltjes,[*] and the proof is based on methods similar to that used in Graeffe's method for determining the roots of an algebraic equation with numerical coefficients.

20. Ruled surfaces with isotropic generators have been considered by Monge, J. A. Serret, Lie, and others. Professor Eisenhart uses for the determination of such a surface the curve in which it is cut by a plane and the directions of the lines which are the projections on the plane of the generators of the surface. In this way a ruled surface of this type is determined by a set of lineal elements, in a plane, depending upon one parameter. The envelope of the direction of the lineal elements is a generalization of oblique evolutes, or evolutoids. Such a surface is determined also by a curve in space, in the sense that it is one of the two sheets of the envelope of a certain family of spheres whose centers are on the space curve. The equations of the two sheets assume a symmetrical form when the space curve is defined in two ways in terms of the two "normal parameters," as defined by the author in the September number of the *Annals*. The relations between the space curve and the set of lineal elements on the plane are very interesting, and lead to a significant transformation of sets of plane curves. Of particular interest is the case in which the spheres are of constant radius, so that the Gaussian curvature of the surface is a positive constant, and thus the ruled surface is a deform of a sphere in which one set of numerical generators remain right lines.

21. In his first paper Professor Young uses the essential ideas of von Staudt's algebra of points on a line to develop the general notion of an algebra associated with certain groups

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[*] Correspondance d'Hermite et de Stieltjes, vol. 2 (1905), p. 185.

of transformations. He considers first a group $\mathfrak{G}$ of transformations which operates in a simply transitive way on the elements a, b, c, $\cdots$ of a certain class Σ. By choosing any particular element of Σ, say i, to be the so-called identical element, there exists in $\mathfrak{G}$ a unique transformation G_a, such that $G_a(i) = a$. An operation o acting on pairs of elements of $\mathfrak{G}$ is then defined by the relation $a \circ b = G_a(b)$. The operation o is then associative; it is commutative if $\mathfrak{G}$ is commutative; the element i satisfies the relations $a \circ i = i \circ a = a$ for every a of Σ; and corresponding to every a in Σ there exist in Σ two unique elements a', a'' such that $a \circ a' = i$ and $a'' \circ a = i$. The group $\mathfrak{G}$ is then represented by the equations $x' = a \circ x$, the individual transformations of $\mathfrak{G}$ being obtained by allowing a to vary over Σ. The equations $x' = x \circ a$ define a group $\overline{\mathfrak{G}}$, simply isomorphic with $\mathfrak{G}$. $\overline{\mathfrak{G}}$, which is called the adjoint group of $\mathfrak{G}$, is identical with $\mathfrak{G}$ only when $\mathfrak{G}$ is commutative. The author considers next two groups, say $\mathfrak{A}$ and $\mathfrak{M}$, acting in a simply transitive way on two classes Σ_1 and Σ_2 respectively. The operations defined as above by $\mathfrak{A}$ and $\mathfrak{M}$ are for convenience called "addition" and "multiplication" and denoted by $+$ and $\cdot$ respectively; the identical element with respect to $+$ is denoted by 0 and the identical element with respect to $\cdot$ by 1. If Σ_1, Σ_2 have a non-empty class Σ in common, there will be relations between addition and multiplication determined by the relations between the groups $\mathfrak{A}$ and $\mathfrak{M}$. If, in particular, $\mathfrak{M}$ transforms $\mathfrak{A}$ into itself, there is obtained a generalized law of distributivity, expressed by the following relation: $a \cdot (b+c) = a \cdot b + (a \cdot 0)' + a \cdot c$, where the notation m' is used for the element such that $m' + m = 0$. The ordinary form of distributivity is obtained if $a \cdot 0 = 0$ for all values of a; i. e., if every transformation of $\mathfrak{M}$ transforms 0 into itself (0 is then not an element of Σ_2; it is exceptional for $\mathfrak{M}$). The distributive relation just given implies the commutativity neither of addition nor of multiplication. The condition $\mathfrak{M}\mathfrak{A}\mathfrak{M}^{-1} = \mathfrak{A}$, leads to general left-handed distributivity. The condition for general right-handed distributivity is $\overline{\mathfrak{M}}\mathfrak{A}\overline{\mathfrak{M}}^{-1} = \mathfrak{A}$, where $\overline{\mathfrak{M}}$ is the adjoint group of $\mathfrak{M}$. Von Staudt's algebra of points on a line is obtained by letting $\mathfrak{A}$ be the group of all parabolic projectivities having a common double point M, and letting $\mathfrak{M}$ be the group of all projectivities having two common double points M and O, where O is the identical element with respect to $\mathfrak{A}$. The author discusses several other examples which lead to new analytic representations of certain groups of projective and birational transformations.

22. In his second paper, Professor Young employs the methods developed in his first paper to define a point algebra in a space S_n of n dimensions as follows: Multiplication is defined by the group $\mathfrak{M}$ of collineations in S_n leaving the vertices $O, U_1, U_2, \cdots, U_n$ of a complete $(n + 1)$-point invariant, $a \cdot b$ being by definition the point into which b is transformed by the transformation M_a of $\mathfrak{M}$ which transforms the unit point 1 into a. Addition is defined by the group $\mathfrak{A}$ of elations in S_n leaving fixed every point of the S_{n-1} determined by $U_1, \cdots, U_n$, $a + b$ being the point into which b is transformed by the transformation A_a of $\mathfrak{A}$ which transforms the zero point $O = 0$ into a. Multiplication and addition are then associative and commutative, and multiplication is distributive with respect to addition. In fact, the points of S_n form a field with respect to these two operations, the exceptional elements being the points of the n $(n - 1)$-spaces of the fundamental $(n+1)$-point $O, U_1, \cdots, U_n$ for multiplication and the points of the S_{n-1} determined by $U_1, \cdots, U_n$ for addition. The author then considers the group $\mathfrak{A}_n$, of $3n$ parameters, consisting of all transformations $x' = \dfrac{ax + b}{cx + d}$, $ad - bc$ not a divisor of zero. $\mathfrak{A}_1$ is then the general projective group on a line; $\mathfrak{A}_2$ is the group of all quadratic transformations in a plane having two fundamental points U_1 and U_2 in common, and is therefore equivalent to the group of direct circular transformations in a plane first discussed by Möbius; $\mathfrak{G}_3$ is a group of Cremona transformations in S_3 of order 3 having three fundamental points U_1, U_2, U_3 in common; in general $\mathfrak{G}_n$ is a group of Cremona transformations of order n in S_n. The representation of the group given above makes it possible to apply methods of the projective geometry on a line to the geometry defined by $\mathfrak{G}_n$. The elementary curves of the geometry, the so-called C-curves, are the curves equivalent under $\mathfrak{G}_n$ to the straight line through 0 and 1. There is one and only one such curve through any 3 points a, b, c, provided no two of the latter are coplanar with two of the points U_i. The class of C-curves contains all the normal

curves of degree n in S_n, passing through the n points U_i ($i = 1, \cdots, n$) and certain normal curves of lower orders. For example, for $n = 3$ the C-curves consist of all twisted cubics through U_1, U_2, U_3, of all conics through one of the points U_i and meeting the opposite side of the triangle $U_1U_2U_3$, and of all straight lines in S_3 not on the plane $U_1U_2U_3$. Any transformations of $\mathfrak{G}_n$ of general type has 2^n double points, which are associated in a definite way into 2^{n-1} pairs. Among the theorems derived may be mentioned the following: Any involution (of general type) in $\mathfrak{G}_n$ leaves invariant every C-curve through every pair of associated double points, and only these. If a transformation of $\mathfrak{G}_n$ leaves one C-curve invariant, it leaves one, and if the transformation is non-involutoric only one, such curve through every point not a double point invariant; etc. The author also discusses the extension of the group $\mathfrak{G}_n$ by the adjunction of transformations of another kind, which reduces in the case $n = 2$ to the introduction of the indirect circular transformations.

23. A series is absolutely convergent, if the series formed from the absolute values of its terms converges; unconditionally convergent, if it converges for every order in which its terms are arranged. The two definitions are equivalent for convergent series. In this note, Dr. Silverman proposes to inquire whether similar notions are possible in the theory of summable series.

The notion of absolute convergence is obviously not generalizable to the case of summable series,* if we restrict† ourselves to those definitions for which it is true that whenever $\lim(S_n) = s$ or ∞, then $G(S_n) = s$ or ∞ respectively, where lim stands for the ordinary limit, and G for the generalized limit. That the notion of unconditional convergence, too, cannot be generalized to the case of summable series, if we restrict ourselves to the same definitions as above, results from the following theorem: If corresponding to every arrangement of the terms of a series there is some definition of summability which gives it a value (possibly different in each case), then the series is absolutely convergent.

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* Borel has used the term absolute summability, but in a different sense from that considered here. See Leçons sur les Séries divergentes, p. 99.

† This restriction excludes such definitions as that of absolute summability given by Borel; but this definition of Borel's is not a true generalization of convergence, as has been pointed out by Hardy, *Quarterly Journal*, vol. 35 (1903), p. 25.

24. In this note Dr. Silverman considers some simple sufficient conditions for Cesàro summability, similar to some of the well-known tests for convergence.

25. Professor Fite calls attention to a collineation group in n variables that seems not to have been noticed before. It is of order $n^3b^{a-1}d^{c-1}/f$, where $n = ab = cd$, a and b relatively prime, c relatively prime with both a and d, $a < n$, $c < n$, and f the greatest common divisor of $c - a$ and d. It contains an invariant abelian subgroup of index n.

26. Professor Eisenhart gives the following analytical definition of a congruence of minimal lines in euclidean four-space: Given a congruence C of non-minimal lines in ordinary space, each line being defined by the coordinates (x_0, y_0, z_0) of the point P_0 in which the line meets a surface of reference S referred to curvilinear coordinates u, v and by its direction cosines X, Y, Z, where x_0, y_0, z_0 and X, Y, Z are one-valued continuous functions of u and v, admitting continuous first derivatives; if x, y, z are the coordinates of a point P on this line and t the distance P_0P, then the equations

$$x = x_0 + tX, \quad y = y_0 + tY, \quad z = z_0 + tZ, \quad w = it$$

define a congruence Γ of minimal lines in four-space of the most general type. The condition that a one-parameter system of these lines, determined by a relation $f(u, v) = 0$, be tangent to a curve, i. e., developable two-spread, is that the corresponding ruled surface of the congruence C, which is defined by the first three of the above equations, be a developable surface, and that it meets S in an orthogonal trajectory of the generators of this developable. When C is a normal congruence, and only in this case, through each line of Γ there pass two developables of the congruence. (This is the particular type of minimal congruences recently considered by Eiesland in the *Transactions*.) However, any surface S and a family of curves upon it leads to a congruence Γ such that through each line of Γ there passes one developable. But for the general congruence Γ none of the ruled two-spreads is developable. The paper deals with other properties of congruence Γ.

F. N. Cole,
Secretary.

THE FIFTH REGULAR MEETING OF THE SOUTHWESTERN SECTION.

THE fifth regular meeting of the Southwestern Section of the Society was held at Washington University, St. Louis, Missouri, on Saturday, December 2, 1911. About twenty-five persons were present, including the following members of the Society:

Professor W. C. Brenke, Dr. A. R. Crathorne, Professor Ellery W. Davis, Dr. Otto Dunkel, Professor H. C. Harvey; Professor E. R. Hedrick, Professor G. O. James, Professor O. D. Kellogg, Professor A. M. Kenyon, Mr. W. C. Krathwohl, Dr. S. Lefschetz, Professor H. L. Rietz, Professor W. H. Roever, Professor J. N. Van der Vries, Professor C. A. Waldo.

The morning session opened at 10 A.M. and the afternoon session at 3 P.M., Professor Roever presiding. Lawrence, Kansas, was decided upon as the next place of meeting and the following program committee was elected: Professors Van der Vries (chairman), Davis, and Kellogg (secretary). The members presentt atended a smoker at the University Club on the evening before the meeting and dined together between the sessions in the dining room of Tower Dormitory.

The following papers were presented at this meeting:

(1) Professor R. D. CARMICHAEL: "Generalizations of Euler's φ-function with applications to abelian groups."

(2) Dr. OTTO DUNKEL: "A necessary condition for the reality of the roots of an algebraic equation."

(3) Professor O. D. KELLOGG: "Note on periodic functions with derivatives of all orders."

(4) Dr. S. LEFSCHETZ: "Two theorems on conics."

(5) Professor M. B. WHITE: "The dependence of the focal point on curvature in space problems in the calculus of variations."

(6) Professor IVA ERNSBERGER: "The imaginary parabola" (preliminary communication).

(7) Professor HEDRICK and Dr. INGOLD: "Axioms of line geometry."

(8) Professor W. H. ROEVER: "The deviation of falling bodies for a distribution not of revolution."

(9) Professor E. R. HEDRICK: "The concept of limit."

(10) Professor G. O. JAMES: "Comparison of methods of adjusting outstanding differences in the motions of the four inner planets."

(11) Dr. A. B. FRIZELL: "On a subset of the terms in the infinite determinant having the potency of the continuum."

Professor Ernsberger was introduced by Professor Davis. In the absence of the authors Professor Carmichael's paper was read by title and the papers of Professor White and Dr. Frizell were presented by Professor Van der Vries. Abstracts of the papers follow.

1. Among the many generalizations which have been given of Euler's φ-function, particular mention may be made of those due to Schemmel, Lucas, Vahlen, Cohen, and Zsigmondy. The last (*Monatshefte für Mathematik und Physik*, volume 7 (1896), pages 185–289) especially, is one of far-reaching importance. It comes out as an application of a general theory of abelian groups, which Zsigmondy develops. The object of Professor Carmichael's paper is to define in a new way Zsigmondy's generalization of the φ-function, and to develop its fundamental properties from the point of view of the new definition. As immediate corollaries of this theory of the generalized φ-function, one obtains Zsigmondy's fundamental results in the theory of abelian groups by means which are essentially simpler than those employed by Zsigmondy.

2. A necessary condition that all the roots of the equation $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$ be real is that the series of determinants of even order

$$\begin{vmatrix} a_0 & a_1 \\ a_1 & a_2 \end{vmatrix}, \qquad \begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \\ a_3 & a_4 & a_5 & a_6 \end{vmatrix}, \quad \cdots$$

have signs alternately $-$ and $+$. This result which is obtained by Dr. Dunkel by the methods employed in a previous paper in the *Annals of Mathematics*, series 2, volume 10 (1908), page 46, bears some analogy to a sufficient condition already given by Professor E. B. Van Vleck in the same journal, series 2, volume 4 (1903), page 191.

3. If $f(x)$ together with its derivatives of all orders has the period 2π, then either $\max|f^{(k)}(x)| > Ae^{\lambda k}$ for all but a finite number of values of k, A and λ being positive constants, or else $f(x)$ is a terminating Fourier series. Professor Kellogg's paper gives a simple proof of this fact, and mentions some inferences.

4. In this paper Dr. Lefschetz treats of two correlated theorems on conics. (1) When two complete quadrilaterals, circumscribed to the same conic S have a common diagonal, their vertices exterior to it are on a conic Γ. Let D be the common diagonal, A, B, A', B' the vertices on it. It is shown both analytically and synthetically that for a given conic the vertices A, B are conjugate elements of an involution on D, and conversely that to each involution on D there corresponds one and only one conic Γ. These theorems can be deduced easily from others given by Poncelet, although the proofs of the present paper are completely different. It is further shown that the system S, Γ is projectively ∞^1, and the relation satisfied by the invariants of the two conics is also given. (2) When the tangents at three of the six common points of a conic and a triangle go through the opposite vertices, they all do, the triangle being then self-polar with respect to the conic. Here also both analytic and synthetic proofs are given, and a third proof based upon projective considerations. This theorem is equivalent to the following well known proposition in the geometry of rational plane quartics: if a rational plane quartic has three flecnodes, it has necessarily three biflecnodes. The paper will be offered for publication to the *Annals of Mathematics*.

5. If an arc C_{01} which joins a space curve L and a fixed point 1 minimizes the integral $J = \int f(x, y, y', z, z')dx$ with respect to other curves joining L with 1, there will in general be a focal point 2 on the curve of which C_{01} is a part at which this minimizing property ceases. Professor White shows that the position of the focal point is a function of the curvature of L at 0 and discusses its properties by means of the second variation. The paper will be offered to the *Transactions*.

7. In this paper Professor Hedrick and Dr. Ingold give a system of axioms for line geometry based upon the fundamental ideas of line and intersection. They are able with a com-

paratively simple set of assumptions, to prove the fundamental propositions of line geometry. The assumptions may also be made the basis of a treatment of projective geometry by defining the words point, plane, etc., in terms of the fundamental ideas mentioned above.

8. In order to be able to compare the results of experiment with those of theory, the following definitions are taken: Let P_0 denote the point from which the body falls, c the path of the falling body, and d the locus of plumb-bobs of all plumb-lines which are supported at P_0. The curve d pierces a level surface in a point D. Let π denote the tangent plane to this level surface at D, and C the point in which the curve c pierces the plane π. Let α and β denote the east-and-west and the north-and-south lines respectively of the plane π which pass through D. The distance of C south of α is the southerly deviation of the falling body and that of C east of β is the easterly deviation of the falling body. If h represents the height of P_0 above π and W the potential function of the field of force in which the plumb-line is in equilibrium, it is possible to express the deviations just defined in terms of h and the first and second derivatives of W at P_0. These expressions are deduced in Professor Roever's paper. The quantities h and g (the only first derivative of W which enters) can be measured. The second derivatives of W which enter can be found experimentally by a method due to Baron Eötvös. (Encyklöpädie der mathematischen Wissenschaften, Band VI, 1, B, Heft 2, § 23, page 166.) Hence it is not necessary to assume a mathematical expression for W. Since experiments have been, and are likely to be, made in localities where the local deviation of the gravitational field is considerable (due to the proximity of mountains) the advantage of this method is apparent.*

9. In the paper presented by Professor Hedrick it is shown that a material simplification of the system of axioms proposed by Steinhaus in a recent number of the *Mathematische Annalen* for the concept "limit" leaves that concept defined uniquely on convergent sequences, and permits extension to divergent sequences in the Cesàro sense. It is shown that the concept

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* The southerly deviation for a distribution of revolution has already been treated by Professor Roever in a paper in the *Transactions*, vol. 12, No. 3, pp. 335–353.

"limit" may be defined for general systems of objects in an analogous manner, without excluding the possibility of special examples of the Cesàro type.

10. The nodes and perihelia of the four inner planets, notably Venus and Mars, present certain unexplained motions in the Newtonian mechanics. The note of Professor James compares the secular changes in the elements of these planets produced by the uniform rotation of the empirical about the inertial system of reference with the corresponding changes brought about by the use of the Minkowskian law of attraction instead of the Newtonian.

11. In this paper Dr. Frizell shows that a one-to-one relation exists between the continuum and a set of terms in the expansion of an infinite determinant whose elements are restricted to the principal diagonal and two adjacent diagonals.

O. D. KELLOGG,
Secretary of the Section.

SERIES OF LAPLACE'S FUNCTIONS.

BY PROFESSOR B. H. CAMP.

(Read before the American Mathematical Society, October 28, 1911.)

THE most important theorem on the validity of the expansion of an arbitrary function in a series of Laplace's functions has been proved by Jordan in his Cours d'Analyse, second edition, volume 2, page 252. The conditions there stated are that the given function be continuous on the surface of the sphere within some small circle about the point at which the expansion is made, and that it have limited variation along every great circle through this point.

The object of the present paper is to correct an error in Jordan's theorem, and to furnish new conditions sufficient for the validity of these expansions. To the conditions announced by Jordan should be added the requirements that the values of the variations be all less than some fixed number, and that these variations be "uniform with respect to all great circles through the point." His error is discussed in a remark following Corollary 2.

In Theorem 1, I have corrected Jordan's theorem and extended it to the case where any or all of the conditions may fail on a null set of great circles through the point. Theorem 2 is a new theorem, and replaces the conditions of continuity of the function and uniformity of the variations, in Theorem 1, by the different requirement that there exist a small circle about the point considered such that within it the given function is, along every great circle through the point, an indefinite integral of another function which has an absolutely convergent double Lebesgue integral in its domain of definition. A null set of great circles may be neglected in this theorem also.

All the integrals used are *L*-integrals, i. e., integrals in the sense of Lebesgue. Limited variation may be understood to refer either to the definition of Jordan or to that of Pierpont; the theorems are true for both definitions.

Suppose the function $f(p)$ to be defined and limited and to have a double Lebesgue integral on the surface of a unit sphere, p being a point of the surface, and suppose it to be required to develop this function in a series of Laplace's functions which shall be valid at a fixed point p_0. We learn from Jordan's work, subject to certain transformations which are permissible* in the present case, that the validity of the formal development depends on the convergence to $\Phi(x_0)$ of

$$S_n = \frac{1}{4\pi}\int_0^{2\pi} d\mu \int_{-1}^{+1} \Phi(x, \mu)\,[X'_{n+1}(x) + X'_n(x)]dx, \tag{1}$$

where $\Phi(x, \mu)$ is the form assumed by $f(p)$ when the coordinates (x, μ) of p are chosen as stated below, and X'_1 is the derivative with respect to x of X_n, the function of Legendre. Here p_0 is taken as the north pole of the sphere, x as the cosine of the colatitude of p, and μ as the longitude of p with respect to some fixed meridian $\mu = 0$. Evidently $f(p_0) = \Phi(x_0) = \Phi(1)$, and is independent of μ. Now since

$$\int_{-1}^{+1} (X'_{n+1} + X'_n)\, dx = 2,$$

it follows that

$$\Phi(x_0) = \frac{1}{4\pi}\int_0^{2\pi} d\mu \int_{-1}^{+1} \Phi(x_0)\,(X'_{n+1} + X'_n)dx.$$

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* Lebesgue, *Annales l'Ecole Norm. Sup.*, ser. 3, vol. 27 (1910), pp. 447–50.

Subtracting this from (1), we have, placing $\psi(x, \mu) = \Phi(x, \mu) - \Phi(x_0)$,

$$(2) \qquad 4\pi(S_n - \Phi(x_0)) = \int_0^{2\pi} d\mu \int_{-1}^{+1} \psi(x, \mu)\,(X'_{n+1} + X'_n)dx,$$

and the validity of the expansion to be studied depends on the convergence of (2) to zero with $1/n$.

LEMMA 1. *If $f(x)$ has limited variation of value V in the interval (a, b), and $\phi(x)$ is absolutely L-integrable in this interval,*

$$\left|\int_a^b f\phi\right| \leq (V + \max|f|)\max_{\lambda}\left|\int_a^{\lambda}\phi\right|, \ \lambda \text{ in } (a, b).$$

This is readily deduced from a theorem of Lebesgue's (*Annales de la Faculté de Toulouse*, series 3, volume 1 (1909), page 37).

THEOREM 1. (1°) *Let $\Phi(x, \mu)$ be defined and limited and have a double* L-integral in the rectangle $R = (-1 \leq x \leq 1, 0 \leq \mu \leq 2\pi)$;* (2°) *let it be, at $x = 1$, a function of x uniformly continuous with respect to μ;* (3°) *let it have limited variation in $(-1, 1)$, and let the value of this variation be limited with respect to μ.* (4°) *Let the variation of Φ in $(1 - x, 1)$ be, at $x = 1$, a function of x uniformly continuous with respect to μ. Finally, any or all of these conditions may fail for a null of set μ's.*

Then the development of $f(p)$ in Laplace's functions is valid at p_0.

Proof. We will establish the theorem by showing that (2) converges to zero with $1/n$. Let ϵ be > 0 and arbitrary, and select h so that by 2°

$$(3) \qquad \max|\psi| < \epsilon \text{ in } (1 - h, 1)$$

for all μ's uniformly, except perhaps a null set; and so that by 4°, for the function ψ,

$$(4) \qquad V(1 - h, 1) < \epsilon$$

for all μ's uniformly, except perhaps a null set. We may

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* It is known that

$$\int_R \psi(X'_{n+1}+X'_n) = \int_0^{2\pi} d\mu \int_{-1}^{+1} \psi(X'_{n+1}+X'_n)dx.$$

A slightly more general condition would suffice, but the difference is trivial.

write (2) in the form

$$(5)\qquad \begin{aligned} &\int_0^{2\pi} d\mu \int_{-1}^{1-h} \psi(X'_{n+1} + X'_n)dx \\ &+ \int_0^{2\pi} d\mu \int_{1-h}^{1} \psi(X'_{n+1}+X'_n)dx = \int_0^{2\pi} (I)\, d\mu + \int_0 (II) d\mu. \end{aligned}$$

By 3° and Lemma 1,

$$|\,\mathrm{II}\,| \leqq (V(1-h, 1) + \max |\,\psi\,|\,) \max_{\lambda} \left| \int_{\lambda}^{1} (X'_{n+1} + X'_n)dx \right|,$$

λ in $(1-h, 1)$, for all μ's, except perhaps a null set. But

$$\left| \int_{\lambda}^{1} (X'_{n+1} + X'_n)dx \right| = |\, 2 - X_{n+1}(\lambda) - X_n(\lambda)\,| < 4,$$

and therefore

$$(6)\qquad |\,\mathrm{II}\,| < 4(\epsilon + \epsilon),$$

by (3) and (4), for all μ's uniformly, except perhaps a null set. Similarly

$$|\,I\,| \leqq (V(-1, 1-h) + \max |\,\psi\,|\,) \max_{\lambda} \int_{-1}^{\lambda} (X'_{n+1} + X'_n)dx,$$

λ in $(-1, 1-h)$, for all μ's except perhaps a null set. The integral here equals $X_{n+1}(\lambda) + X_n(\lambda) - 0$, and Jordan has shown (page 236) that this expression converges to zero with $1/n$ uniformly with respect to λ in $(-1, 1-h)$, provided only that $0 < h \leqq 2$. Since, by 1° and 3°, there exists an M such that $V + \max |\,\psi\,| < M$, we have now shown that there exists an n_ϵ, independent of μ, so that

$$(7)\qquad |\,I\,| < M\epsilon, \quad n > n_\epsilon,$$

for all μ's uniformly except perhaps a null set. This with (6) and (5) shows that

$$\left| \int_0^{2\pi} d\mu \int_{-1}^{+1} \psi\, (X'_{n+1} + X_n')dx \right| < 2\pi(8\epsilon + M\epsilon), \quad n > n_\epsilon;$$

which proves the theorem.

Remark. *In order that* 4° *may be fulfilled, it is sufficient* *that* Φ *be, with respect to* μ, *uniformly continuous at* $x = 1$, *and that in* $(-1, 1)$ *it have limited variation which is "uniform with respect to* μ" (*i. e., that* $\lim_{d=0} V_D = V$ *uniformly with respect to* μ, *where* V *is the value of the variation in* $(-1, 1)$, *and* V_D *the usual sum function for the arbitrary division* D *of norm* d).

COROLLARY 1. *The same conclusion follows if, in place of* 3° *and* 4°, *we have the condition that* Φ *be a monotone increasing function of* x.

For 3° is then satisfied because $V \leq \max |\psi|$ in the rectangle R. Moreover, if $d \leq 2$, $V - V_D = 0$, for all μ's, except perhaps a null set. Therefore, by the Remark above, 4° is also satisfied.

COROLLARY 2. *If* $f(p) = f_1(p) - f_2(p)$, *where* f_1 *and* f_2 *both satisfy the conditions of Corollary* 1, *the same conclusion follows.*

It is this last corollary that Jordan has actually proved, except that he may not neglect a null set of meridians, and that he should have stated that f must be limited on the sphere as well as along each meridian. (Cf. page 230, "D'autre part $\psi(b - 0)$ est fini. . . ." At the bottom of page 251 he desires to have the convergence uniform with respect to μ, and therefore ψ "fini" uniformly with respect to μ.) Having proved this corollary, however, it is not permissible to conclude, as Jordan does, that the theorem will be true without the restrictions stated in 4° and in the last part of 3°. As he remarks in § 220, his theorems are proved for monotone functions, and stated for functions of limited variation. This is a proceeding which is often allowable, because of the well known fact that any function of limited variation is the difference of two monotone increasing functions, but it may lead to error unless the two monotone functions may be made to obey the restrictions imposed on the given function. As a matter of fact, in the case before us, it is not difficult to construct a limited, integrable function, which satisfies the conditions imposed by Jordan, but is not expressible as the difference of two monotone increasing functions each of which separately obeys the same conditions. In a remark appended to a fundamental convergence theorem (§ 222, page 230) Jordan recognizes this difficulty, but he ignores his remark when he afterwards applies his theorem to Laplace's functions. He has actually found, then, the conditions which certain components of the given function should satisfy, not those which are to be satisfied by the function itself.

LEMMA 2. *If of two functions, of one or of two variables, one is absolutely L-integrable in a limited field, and the other is limited, their product is absolutely L-integrable in the same field.**

LEMMA 3. *Let A be any limited, measurable field in the x, y plane. In order that the integral*

$$\int_A f(x, y)\,\phi(x, y;\, n)$$

may converge to zero with $1/n$ for all absolutely L-integrable functions f defined in A, it is necessary and sufficient that there exist an M and an n_M so that $|\,\phi\,| < M$, $n > n_M$, except perhaps at a null set of points, and that $\int_r \phi \doteq 0$ with $1/n$ for each rectangle r of sides parallel to the axes x, y.

This may be proved in a manner analogous to the proof given by Lebesgue for the corresponding theorem in one dimension (*Annales de la Faculté de Toulouse*, loc. cit., page 52)† by the use of another theorem proved by him in *Annales de l'Ecole Normale Supérieure*, series 3, volume 27 (1910), page 374.

THEOREM 2. *Using the notation of Theorem* 1, (1°) *let $\Phi(x, \mu)$ be defined, limited, and have a double L-integral in R.* (2°) *In $(1 - h, 1)$, for some $h > 0$ and independent of μ, let Φ be the indefinite integral, with respect to x, of another function $\theta(x, \mu)$, which has an absolutely convergent, double L-integral in its field of definition.* (3°) *Let Φ have limited variation in $(-1, 1)$ the value of which is limited with respect to μ. Finally, any or all of these conditions may fail for a null set of μ's.*

Then the development of $f(p)$ in Laplace's functions is valid at p_0.

Proof. As in Theorem 1, we may write (2) in the form (5), and, by virtue of 1° and 3° of our hypothesis, the first of these integrals,

$$(8) \qquad \left|\int_0^{2\pi} d\mu \int_{-1}^{1-h} \psi(X'_{n+1} + X'_n)dx\right| < 2\pi M\epsilon, \quad n > n_\epsilon.$$

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* Lebesgue, loc. cit., p. 374.

† Cf. also Hobson, *Proc. London Math. Soc.*, ser. 2, vol. 6 (1908), p. 355.

As for the second, by 2° and a theorem due to Lebesgue,*

$$(9)\qquad \int_{1-h}^{1}\psi(X'_{n+1}+X'_n)dx = [\psi(X_{n+1}+X_n)]_{1-h}^{1}$$

$$-\int_{1-h}^{1}(X_{n+1}+X_n)\theta dx = -\psi(1-h,\mu)\cdot(X_{n+1}(1-h)+X_n(1-h))$$

$$-\int_{1-h}^{1}(X_{n+1}+X_n)\theta dx, \quad \text{since } \psi(1,\mu)=0.$$

We may now write the second integral of (5) in the form

$$(10)\ \int_0^{2\pi}d\mu\int_{1-h}^{1}\psi(X'_{n+1}+X'_n)dx = -\int_0^{2\pi}d\mu\int_{1-h}^{1}\psi(1-h,\mu)$$

$$\cdot(X_{n+1}(1-h)+X_n(1-h))dx-\int_0^{2\pi}d\mu\int_{1-h}^{1}(X_{n+1}+X_n)\theta dx,$$

provided two of these integrals exist. To show that they do exist let us write R_h for the rectangle $1-h\leq x\leq 1$, $0<\mu\leq 2\pi$. Then, by 2° and Lemma 2, $\int_{R_h}(X_{n+1}+X_n)\theta$ exists, and by a recent theorem of Hobson†

$$(11)\qquad \int_{R_h}(X_{n+1}+X_n)\theta = \int_0^{2\pi}d\mu\int_{1-h}^{1}(X_{n+1}+X_n)\theta dx.$$

We knew before that the first integral of (10) exists.

We now employ again that portion of Jordan's work used to derive (7) in Theorem 1, and learn that

$$|X_{n+1}(1-h)+X_n(1-h)|<\epsilon, \quad \text{if } n>n_\epsilon,$$

and by 1° we know that $|\psi(1-h,\mu)|<M$; and these inequalities hold for all μ's uniformly, except perhaps a null set. By virtue of these and (11), (10) shows us that

$$(12)\ \left|\int_0^{2\pi}d\mu\int_{1-h}^{1}\psi(X'_{n+1}+X'_n)dx\right|<2\pi\epsilon M+\left|\int_{R_h}(X_{n+1}+X_n)\theta\right|,$$

if $n>n_\epsilon$. Comparing this with (8) we find that in order to establish the theorem it only remains to show that

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* Loc. cit., p. 46.

† *Proc. Lond. Math. Soc.*, ser. 2, vol. 8 (1910), p. 30.

$$(13) \qquad \int_{R_k} (X_{n+1} + X_n)\theta \doteq 0 \quad \text{with} \quad 1/n.$$

To do this we use Lemma 3, θ being an absolutely L-integrable function by hypothesis, and $| X_{n+1} + X_n |$ being ≤ 2 for all values of n.

Let r be an arbitrary rectangle in R_k whose sides are parallel to the μ, x axes. It only remains to show that

$$(14) \qquad \int_r X_{n+1} + X_n) \doteq 0 \quad \text{with} \quad 1/n.$$

By two well known formulas,* if $| k | < 1$,

$$\int_k^1 X_n(x)dx = \frac{(1-k^2)(X_n'(k))}{n(n+1)},$$

and

$$X_n'(k) = \frac{nkX_n(k) - nX_{n-1}(k)}{k^2 - 1}.$$

Therefore

$$\left| \int_k^1 X_n(x)dx \right| < \frac{k+1}{n+1}, \quad \text{since} \quad | X_n(k) | < 1.$$

This integral does not depend on μ, and k depends only on r. (14) follows, and the theorem is proved.

The necessary and sufficient conditions that a function be an indefinite integral are given by Lebesgue in the *Rendiconti dell' Accademia dei Lincei*, volume 16 (1907), 1st semester.

The following function satisfies all the conditions of Theorem 2, but not 2° of Theorem 1:

$$\begin{aligned} \Phi &= \frac{1}{\sqrt{\mu}}(x-1) \quad \text{in} \quad 1 - \frac{\sqrt{\mu}}{3} \leq x \leq 1, \text{ if } \mu \neq 0 \\ &= -\frac{1}{3} \text{ elsewhere, if } \mu \neq 0 \\ &= 0, \text{ if } \mu = 0. \end{aligned}$$

WESLEYAN UNIVERSITY,
October, 1911.

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[*] Cf., e. g., Byerly, Fourier's series and spherical harmonics, pp. 172, 180.

ON A NEW MIXED PROBLEM OF THE PARTIAL DIFFERENTIAL EQUATION OF TELEGRAPHY.

BY PROFESSOR ARTHUR GORDON WEBSTER.

(Read before the American Mathematical Society, September 13, 1911.)

DURING the last twenty years the question of the propagation of electric waves in wire lines or cables has been the subject of a very great number of researches, on the one hand on account of its practical importance in connection with telephony, rapid telegraphy, wireless telegraphy, and the transmission of energy by means of alternating currents, and on the other on account of its purely mathematical interest and of the great increase in our knowledge of partial differential equations of the hyperbolic type. In all these researches the point of view has been very different in the different cases and may be roughly classified as follows. The electrical engineer, whether interested in transmission, telephony, or wireless, has been mainly concerned with phenomena depending upon simple harmonic functions of the time, that is, with standing wave phenomena after the steady state of oscillation has been established. For these the methods of trigonometric series suffice perfectly, and with them we are not here concerned. In these cases the line is of finite length, and we deal with series of functions. On the other hand the mathematician has been interested in problems connected with propagation in a line of infinite length, especially as depending on the data of the initial state of the line. For this case instead of infinite series we should use definite integrals of Fourier, but the method of Riemann has furnished a more powerful and interesting method of attack, which should interest the physicist on account of the clearness with which the results may be interpreted. It would be impossible to cite all the important memoirs that have appeared on this subject, but it would be equally impossible to omit the names of Heaviside, Picard, Poincaré, Boussinesq, Goursat, Brillouin, and Hadamard.

The first appearance of the so-called telegraphist's equation is in a paper by Kirchhoff,* which is remarkable in that it appeared before Maxwell's theory of electricity was invented. Twenty years later the equation was first given on the basis of that theory by Heaviside, in a very remarkable paper,* which, like Kirchhoff's, remained unnoticed. Both Kirchhoff and Heaviside used the method of Fourier's series of what are now called normal functions. The treatment of the infinite line began with Poincaré,† who employed the Fourier integral, Boussinesq,‡ who gave a new modification of the method used by Poisson for the wave equation, and Picard,§ who used the method of Riemann.

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* Kirchhoff, "Ueber die Bewegung der Elektricitat in Drahten," *Pogg. Ann.*, vol. 100 (1857); Ges. Abh., p. 131.

All these writers were interested in the propagation of the waves as determined by the initial data in the whole line, that is, in the so-called Cauchy problem. Later appeared papers by Brillouin,|| in which propagation in three dimensions, and the effect of sources of waves was discussed. But in none of these papers was the problem touched which is often of great importance to the experimental physicist, namely the effect upon the waves of the terminal apparatus, composed of any arrangement of coils, condensers, and resistances, which are connected to one or both ends of a line of finite length, and to which is applied at some point an electromotive force which varies according to a given function of the time. It is true that the influence of the terminal apparatus was completely discussed by Heaviside¶ in a very powerful paper written in 1882, but published first in 1892, and in a series of others, well characterized by M. Brillouin, as "une prodigieuse série de mémoires à la fois condensés et touffus," in which the method is throughout that of development in series of normal functions. It is for this reason that Heaviside remarks: "it is an enormous and endless subject, admitting of infinite development." It is believed that for this reason anything that will throw light upon or simplify the matter will be of interest.

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* Heaviside, "On the extra current," *Phil. Mag.*, Aug., 1876; Electrical Papers, vol. I, p. 53

† Poincaré, "Sur la propagation de l'électricité," *Comptes Rendus*, vol. 117 (1893), p. 1027.

‡ Boussinesq, "Intégration de l'équation du son pour un fluide indéfini," *Comptes Rendus*, vol. 118 (1894), p. 162.

§ Picard, "Sur l'équation aux dérivées partielles qui se rencontre dans la théorie de la propagation de l'électricité," *Comptes Rendus*, vol. 118 (1894), p. 16.

|| Brillouin, "Propagation dans les milieux conducteurs," *Comptes Rendus*, vol. 136 (1903), p. 667, 16 mars; p. 746, 23 mars.

¶ Heaviside, "Contributions to the theory of the propagation of current in wires," Electrical Papers, vol. 1, p. 141. "On the self-induction of wires, *Phil. Mag.*, 1886–7; Papers, vol. 2, p. 168.

If I denote the current, V the potential at a point at a distance z from the origin at the time t, the equations of propagation are

$$(1)\qquad L\frac{\partial I}{\partial t}+RI=-\frac{\partial V}{\partial z},\qquad K\frac{\partial V}{\partial t}+SV=-\frac{\partial I}{\partial z},$$

where L is the self-inductance, K the capacity, R the resistance, and S the leakage conductance of the line, all referred to unit of length. If we change the units of length and potential, and remove an exponential factor, putting

$$(2)\qquad \begin{aligned} I&=e^{-qt}u, & V&=-\sqrt{\frac{L}{K}}e^{-qt}v, & 2q&=\frac{R}{L}+\frac{S}{K},\\ x&=\sqrt{KL}\cdot z, & y&=t, & 2p&=\frac{R}{L}-\frac{S}{K},\end{aligned}$$

we reduce to the simple form

$$(3)\qquad \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}-pv,\qquad \frac{\partial v}{\partial x}=\frac{\partial u}{\partial y}+pu,$$

from which we obtain at once

$$(4)\qquad \frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}+p^2u=0,$$

which equation is also satisfied by v. This is the telegraphist's equation which by the substitution $px=\xi+\eta$, $py=\xi-\eta$ is reduced to the normal form

$$(5)\qquad \frac{\partial^2 u}{\partial\xi\partial y}+u=0.$$

When $p=0$ we have the simple equation for the motion of a stretched string

$$(6)\qquad \frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}=0.$$

This, called by Heaviside the distortionless case, was that treated by Kirchhoff. The researches of Picard, Goursat, and Hadamard have been on the more general equation

$$(7)\qquad \frac{\partial^2 u}{\partial x\partial y}+a\frac{\partial u}{\partial x}+b\frac{\partial u}{\partial y}+cu=0,$$

to which most of the results here reached will apply.

The ordinary, or Cauchy problem, consists in determining the solution by giving the values of u and $\partial u/\partial y$ for the whole line $y = 0$. It has been shown by Darboux, Goursat, and Picard, that instead we may give the values of u alone, not giving the normal derivative, along *two* lines meeting at an angle.* It has been remarked by Hadamard† this is not the problem that interests the physicist, but that, as in the present case, we have Cauchy's data for only a portion of the line $y = 0$, while at the end $x = 0$ we have a different condition, not being able to give both Cauchy data, but only one, namely the values of u. This he terms the *mixed* problem.

We are here concerned with a line infinite in one direction‡ which ends at the point $x = 0$, but instead of giving either u or its normal derivative along the line $x = 0$, $y \geqq 0$, we give a relation between them in the form of a linear differential equation in y. It may be easily shown that if we have any combination of coils, with self or mutual inductance and resistance, with condensers, connected to the end of the line, with an electromotive force applied somewhere in the combination, the effect will be to make

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\begin{aligned}
(8)\qquad & a_0 u + a_1 \frac{\partial u}{\partial y} + a_2 \frac{\partial^2 u}{\partial y^2} + \cdots + a_m \frac{\partial^m u}{\partial y^m} \\
& + b_0 v + b_1 \frac{\partial v}{\partial y} + \cdots + b_n \frac{\partial^n v}{\partial y^n} = \psi(y), \quad (x = 0,\ y \geqq 0),
\end{aligned}
$$

from which, eliminating v by the first of equations (3) and an equation (18) or (33), deduced below, we get a differential relation between u and the normal derivative $\partial u/\partial x$. Speaking mechanically, the usual end conditions have been $u = 0$ for a string fixed at the end, or $\partial u/\partial x = 0$ or $\partial u/\partial x - hu = 0$ which have been used for other problems. If the string is fastened, however, to a movable point on an apparatus possessed of elasticity, inertia, and dissipation, the condition will be as stated in (8). The writer was led to consider this condition in 1893, from electrical experiments that he was then making, but on account of lack of mathematical knowledge, the question was laid aside. It appears that the method used by Hadamard leads without difficulty to the result.

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* See also Mason, New Haven Mathematical Colloquium.

† Hadamard, "Sur un probleme mixte aux dérivées partielles," *Bull. de la Soc. Math. de France*, vol. 31 (1903), p. 208.

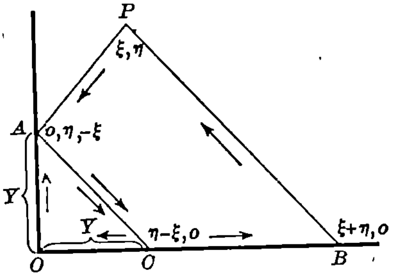
‡ If the second end is at a finite distance, the reasoning here given may be applied to it.

It seems intuitively evident to the physicist that in the present case neither u nor the normal derivative can be given for $x = 0$, but that both are completely determined by the condition (8). The calculation shows this to be the case. For simplicity let us first consider the case of no distortion, $p = 0$. We have then, if we give the initial distribution of current and potential

$$(9) \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}, \qquad \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0,$$

$$(10) \quad u = f(x), \quad v = g(x), \quad \frac{\partial u}{\partial y} = g'(x), \quad y = 0, \quad x \geqq 0.$$

Let P be any point with coordinates ξ, η from which two characteristics meet the axes in A, B, and from A draw the



characteristic meeting the x-axis in C. The fundamental equation of Riemann's method is

$$(11) \quad \int G\left\{\frac{\partial u}{\partial x}dy + \frac{\partial u}{\partial y}dx\right\} - u\left\{\frac{\partial G}{\partial x}dy + \frac{\partial G}{\partial y}dx\right\} = 0$$

for a closed contour, where u and G are solutions of the given equation and its adjoint respectively (here the same equation). In the present simple case the function G is constant and equal to unity, so that we have

$$(12) \qquad \int \left\{\frac{\partial u}{\partial x}dy + \frac{\partial u}{\partial y}dx\right\} = 0.$$

Applying (12) to the closed quadrilateral $PACBP$, we have

$$\begin{array}{llll} & PA, & dx = dy, & \int_P^A du = u_A - u_P, \\ (13) & \left.\begin{array}{l} BP \\ AC \end{array}\right\}, & dx = - dy, & -\int_B^P du = u_B - u_P, \\ & & & -\int_A^C du = u_A - u_C, \\ & CB, & dy = 0, & \int_C^B \frac{\partial u}{dy} dx = \int_C^B \frac{\partial v}{\partial x} dx = v_B - v_C. \end{array}$$

Adding these results

$$(14) \qquad 2u_P = 2u_A + u_B - u_C + v_B - v_C.$$

Accordingly it is necessary to determine only the value of u_A, when we have the result as in Hadamard's case.

If we apply the formula (12) to the triangle AOC, we have

$$(15) \qquad 0 = \int_O^A \frac{\partial u}{\partial x} dy + u_A - u_C + v_O - v_C,$$

so that

$$(16) \qquad u_A = u_C + v_C - v_O - \int_O^Y \left(\frac{\partial u}{\partial x}\right)_{x=0} dy.$$

Inserting in (14), we have

$$(17) \quad u_P = \frac{1}{2}(u_B + u_C) + \frac{1}{2}(v_B + v_C) - v_O - \int_O^Y \left(\frac{\partial u}{\partial x}\right)_{x=0} dy,$$

so that the problem is merely shifted to determining the normal derivative $\partial u/\partial x$ along $x = 0$. Differentiating (16) by Y,

$$(18) \qquad \frac{\partial u(Y)}{\partial Y} = f'(Y) + g'(Y) - \left(\frac{\partial u}{\partial x}\right)_{x=0}.$$

Differentiating the equation (8) and substituting for $\partial v/\partial y$ its value

$$(19) \qquad \frac{\partial u}{\partial x} = f'(y) + g'(y) - \frac{\partial u}{\partial y},$$

we have the linear differential equation

$$(20)\quad (a_0 - b_0)\frac{\partial u}{\partial y} + (a_1 - b_1)\frac{\partial^2 u}{\partial y^2} + \cdots = \psi'(y) - b_0(f'(y) + g'(y)) - b_1(f''(y) + g''(y)) + \cdots.$$

Solving this with the proper initial conditions given by the state of the line and the terminal apparatus at the start completely determines u_A, so that (14) gives the desired solution. This simple result could have been obtained in a perfectly elementary manner from the solution

$$u = F_1(x + y) + F_2(x - y)$$

or by the method of the Fourier integral.

As a very simple example, let us consider the case of a line initially free from current and potential to which a constant electromotive force E is suddenly applied through a coil of inductance L_0 and resistance R_0. Then we have

$$(21)\qquad x = 0,\quad L_0\frac{\partial I}{\partial t} + R_0 I + V = E,$$

or

$$L_0\frac{\partial u}{\partial y} + (R_0 - L_0 q)u - \sqrt{\frac{L}{K}}v = Ee^{qy}$$

and the solution is

$$(22)\quad \begin{aligned} &I = 0, \quad x > t/\sqrt{KL};\\ &I = \frac{E}{R_0 + \sqrt{L/K}}\{1 - e^{-(R_0 + \sqrt{L/K})(t - x\sqrt{KL})/L_0}\} \quad (0 \leqq x \leqq t/\sqrt{KL}). \end{aligned}$$

We thus see the discontinuity at the wave front, and we find that the line reacts on the terminal system like an extra resistance equal to the inductance of the length of line traversed by the wave in unit of time.

Let us now pass to the case of an actual line $p \neq 0$.

$$(23)\qquad \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + p^2 u = 0,$$

$$(24)\qquad u = f(x),\quad \frac{\partial u}{\partial y} = h(x),\quad y = 0,\quad x \geqq 0.$$

The so-called Riemann function G is not now constant, but is such a solution as is equal to the constant unity along the characteristics PA, PB, and accordingly depends not only on the coordinates x, y, but on ξ, η, the coordinates of P. We will therefore write $G(x, y, \xi, \eta)$. If we should integrate our formula (11) around $PACBP$ we should now obtain the integral along the line AC, involving values of u which are unknown. To avoid this we integrate along the quadrilateral $PAOBP$, obtaining

$$\begin{array}{ll} & PA, \quad \int_P^A Gdu - udG = (uG)_A - (uG)_P = u_A - u_P, \\ (24) & BP, \quad \int_B^P udG - Gdu = (uG)_B - (uG)_P = u_B - u_P, \\ & AO, \quad \int_A^O \left(G\frac{\partial u}{\partial x} - u\frac{\partial G}{\partial x}\right)dy, \\ & OB, \quad \int_O^B \left(G\frac{\partial u}{\partial y} - u\frac{\partial G}{\partial y}\right)dx = \int_O^B \left(Gh(x) - f(x)\frac{\partial G}{\partial y}\right)dx, \end{array}$$

$$(25) \qquad \begin{aligned} 2u_P = u_A + u_B + \int_O^B \left(Gh(x) - f(x)\frac{\partial G}{\partial y}\right)dx \\ + \int_O^A \left(u\frac{\partial G}{\partial x} - G\frac{\partial u}{\partial x}\right)dy. \end{aligned}$$

As before, we have to obtain the values of u, $\partial u/\partial x$ along OA. Proceeding as before, we apply our formula to the triangle OAC, but as we wish the Riemann function to be constant along AC we do not use the same G as before, but consider P moved to A, so that we put $G_1 = G(x, y;\ 0, Y)$. We thus obtain

$$(26) \qquad \begin{aligned} u_A = u_C + \int_O^C \left(G_1 h(x) - f(x)\frac{\partial G_1}{\partial y}\right)dx \\ - \int_O^A \left(G_1\frac{\partial u}{\partial x} - u\frac{\partial G_1}{\partial x}\right)dy. \end{aligned}$$

We have now to put, according to Picard,

$$(27) \qquad G(x, y;\ \xi, \eta) = j\{p^2[(y-\eta)^2 - (x-\xi)^2]\},$$

where $j(x) = J_0(\sqrt{-x})$, J_0 representing the Bessel function of order zero. We easily find the values

$$\frac{\partial G}{\partial x}=2p^2(\xi-x)j'\{p^2[(y-\eta)^2-(x-\xi)^2]\},$$

$$\frac{\partial G}{\partial y}=2p^2(y-\eta)j'\{p^2[(y-\eta)^2-(x-\xi)^2]\},$$

$$(28)\quad OB,\ G(x,\,0;\ \xi,\,\eta)=j\{p^2[\eta^2-(x-\xi)^2]\},$$

$$\frac{\partial G}{\partial y}=-2p^2\eta j'\{p^2[\eta^2-(x-\xi)^2]\},$$

$$OC,\ G_1(x,\,0;\ 0,\,Y)=j\{p^2[Y^2-x^2)]\},$$

$$\frac{\partial G_1}{\partial y}=-2p^2Yj'\{p^2[Y^2-x^2]\},$$

$$OA,\quad G(0,\,y;\ \xi,\,\eta)=j\{p^2[(y-\eta)^2-\xi^2]\},$$

$$\frac{\partial G}{\partial x}=2p^2\xi j'\{p^2[(y-\eta)^2-\xi^2]\},$$

$$G_1(0,\,y;\ 0,\,Y)=j\{p^2[(y-Y)^2]\},\quad \frac{\partial G_1}{\partial x}=0.$$

Inserting these in (26)

$$(29)\quad u_A=u_O+\int_0^Y(h(x)j\{p^2[Y^2-x^2]\}+2p^2f(x)Yj'\{p^2[Y^2-x^2]\})dx$$
$$-\int_0^A j\{p^2[(y-Y)^2\}\frac{\partial u}{\partial x}dy.$$

All the functions in the first integrand being known, the integral is a known function of its limit Y, so that (29) is an integral equation between u and $\partial u/\partial x$ of the form

$$(30)\qquad u(y)=\varphi(y)-\int_0^y K(x,\,y)F(x)dx,$$

where the kernel is symmetrical,

$$(31)\quad K(x,\,y)=j\{p^2[x-y]^2\},\quad K(y,\,y)=1,\quad F(y)=\left(\frac{\partial u}{\partial x}\right)_{x=0},$$

reducing to (16) when $p=0$.

Differentiating successively by y, we obtain

$$\frac{\partial u}{\partial y} = \varphi'(y) - F(y) - \int_0^y \frac{\partial K}{\partial y} F(x)dx,$$

$$(32) \qquad \frac{\partial^2 u}{\partial y^2} = \varphi''(y) - F'(y) - \left(\frac{\partial K}{\partial y}\right)_{x=y} F(y) - \int_0^y \frac{\partial^2 K}{\partial y^2} F(x)dx,$$

. . . .

From the differential equation (3) by integration we have

$$(33) \qquad \frac{\partial v}{\partial y} - pv = F(y), \qquad v = \int_0^y e^{p(y-x)} F(x)dx + v_0,$$

from which we may obtain the derivatives of v linearly in terms of those of F and an integral. Substituting all derivatives in the boundary condition (8) we have

$$A_0F(y) + A_1F'(y) + A_2F''(y) + \cdots + \int_0^y \Phi(x, y)F(x)dx = \Psi(y),$$

which is a linear integro-differential equation of Volterra's form.* Solving this for $F(y)$ and inserting in (29) we obtain the desired solution from (25).

We thus see that the question of the reflection of waves at any terminal apparatus is contained in the values of u and $\partial u/\partial x$ along the line OA.

After the preceding paper was completed, I succeeded in seeing Goursat's paper in the *Annales de l'Université de Toulouse*, volume 6 (1904), page 117, and found that he had solved the case of u given along two lines making an acute angle, by means of a Volterra *integral* equation, which is a particular case of the above solution.†

CLARK UNIVERSITY, WORCESTER,
July 5, 1911.

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* *Rendiconti dell' Acc. dei Lincei*, 1909, 1910.

† An abstract of the above paper was presented in the *Comptes Rendus*, Aug. 28, 1911.

NON-EUCLIDEAN GEOMETRY.

Bibliography of Non-Euclidean Geometry including the Theory of Parallels, the Foundations of Geometry, and Space of n Dimensions. By DUNCAN M. Y. SOMMERVILLE. London, Harrison and Sons, 1911. xii+404 pages.

ALMOST simultaneously with its quincentenary celebration, the University of St. Andrews has published this valuable compilation, the result of nine years of research on the part of one of its lecturers in mathematics. The title inevitably suggests similar lists by Halsted, Stäckel and Engel, Schlegel, and the lamented Bonola. To the work of all these bibliographic forebears Dr. Sommerville gives scrupulously exact reference. The bibliography of Halsted, "a model of its kind," includes nearly 200 titles of works relating to non-euclidean geometry and space of n dimensions, from about 1830 to 1879. Short notes appended to the chief works "form a valuable feature." Stäckel and Engel's bibliography on the theory of parallels is a chronological list of nearly 300 titles from 1482 to 1837 and, as Dr. Sommerville remarks, is almost complete. Schlegel's list of works on n dimensions accompanies a report on the subject, which supplies the place of Halsted's notes. It contains about 400 titles arranged alphabetically under the authors. The bibliography of Bonola is the most extensive which has recently appeared. It contains over 900 titles chronologically arranged from 1839 to 1902 with an index of authors. There is a classification under the headings Elementary geometry, Metrical and differential geometry, Group theory, Projective geometry, Finite distances, Vectorial methods, Mechanics and physics, General expositions, Philosophy and history, but there is no subject index.

Dr. Sommerville's idea at first was to prepare a continuation of Halsted's work but the growth of the subject rendered such diffuse treatment impossible and he was led to "produce as far as possible a complete repository of the titles of all works, from the earliest times up to the present, which deal with the extended conception of space, and to form a guide to the literature in an easily accessible form. It includes the theory of parallels, non-euclidean geometry, the foundations of geometry and space of n dimensions. Works on the foundations of mathematics or arithmetic are excluded except in so far as they have an explicit reference to geometry."

The titles of which Dr. Sommerville takes account are over 4,000 in number. They may be roughly classified as follows: theory of parallels 700, non-euclidean geometry and the foundations of geometry 1,600, n dimensions 1,800. The approximate analysis of the nationalities of the authors (excluding mere reviewers and translators) gives: German, 460; French, 230; Italian, 220; British, 160; American, 90; Russian and Polish, 80; Dutch, 35; Hungarian, Bohemian, Croatian, etc., 35; Scandinavian, 35; no attempt is made to distinguish between German and Austrian or between Frenchman, Belgian, and Swiss. The total number of titles in the German language is 1,159; in French 884, Italian 848, English 723, while other languages represented by less than 100 titles are: Russian 99, Latin 80; Dutch 79, Magyar 37, Spanish and Portuguese 27, Danish and Norwegian 27, Polish 22, Swedish 11, Croatian 9, Greek 9, Czech 6, Arabic 4, Esperanto 4, Ruthenian 2, Roumanian 1, Sanskrit 1. The fact that no less than 1,470 titles are assigned to the decade 1901–1910 is an interesting indication of the trend of mathematical research at that time.

Riccardi's Bibliografia Euclidea was the model followed by Dr. Sommerville in the division of his work into three parts: I (pages 1–260) Chronological Catalogue, which covers the period from the time of Aristotle and Plato, who in the fourth century B.C. discussed the parallel postulate and the definitions of point and line, down to June, 1911, although no general attempt was made to obtain titles of later date than 1910; II (pages 251–316) Subject Catalogue in which the classification of the 1908 edition of the Index du répertoire bibliographique is used in amplified form; III (pages 317–398) Author's Index which, as far as possible, gives full names together with year of birth and, in the case of deceased authors, of death. Under each author's name is an abbreviated title and date of each of his writings, the details concerning which are to be found in I.* In the chronological catalogue the titles in each year are arranged alphabetically according to the authors. The various editions, translations and reviews * of a work as well as its classification (II) and price, when a separate work, are here found. References to bibliographical sources have only been given in cases of discrepancies among the authorities. A very few titles are marked with an asterisk (*) as having been personally verified by the author. This fact naturally inspires the fear that the subject classification will require considerable emendation to make it an accurate epitome of the work done in the various fields. On pages 399–404 are given "additions" and "corrections."

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* The seven persons who have the greatest number of titles (including biographical sketches and reviews) to their credit in Dr. Sommerville's list are, in order: Halsted (77), Mansion, Schoute, Segre, Poincaré, Klein, Bianchi.

It may be asked, to what extent has the author succeeded in making a complete work within the prescribed limits? On looking through the list of his sources of information we find beside the bibliographies mentioned above, the *Jahrbuch*; *Revue Semestrielle*; International Catalogue (A); Royal Society Catalogue, Authors, volumes 1–12, Subject, volume 1;† Poggendorff; and Wölffing's Mathematischer Bücherschatz. It is with some surprise that we learn that no use was made of the Royal Society Catalogue, Subject Index, volume 2, "Mechanics," which would have supplied at least four additional titles of value, or of the International Catalogue, (B) Mechanics, (C) Physics, or of the 1,900 slips of the Répertoire bibliographique des Sciences mathématiques. Nor do we anywhere find references to Felix Müller's excellent Führer durch die mathematische Literatur with its well arranged geometry bibliographies which cover many pages. We are therefore prepared for numerous omissions (the more, when we recall the well known imperfections of most of the above mentioned works) and it is indeed probable that they number many hundreds, even though Dr. Sommerville has already digested a great mass of material. A very brief examination of his book suggested a score of additions:‡ D. Lardner, Elements of Euclid, third edition, London, 1832; sixth edition, 1838; eighth, 1843; ninth, 1846; tenth, 1849; W. C. Hume, A treatise on the theory of parallel lines, Dublin, 1853, pages viii + 40 and 2 plates; E. Padova [extension to n-space of two theorems by Neumann on potential], *Giornale di Matematiche*, Napoli, volume 8 (1870), pages 296–301; J. König [Traité analytique des hypothèses de la géométrie], *Értekezések a Mathematikai Osztály Koréböl. Kiadja a Magyar Tudományos Akadémia*, Budapest, volume 6 (1872), pages 94–98; A. Tonelli [Potential function in n-space], *Göttinger Nachrichten*, 1875, pages 521–552; É. Lemmi, "Sur les cas d'exception au théorème des forces vives . . .," *Journal de Mathématiques pures et appliquées*, Paris, volume 2 (1876), pages 233–239; Réthy [Trigonometry in non-euclidean space], *A Magyar Tudományos Akadémia Ertesítoje*, Budapest, volume 25 (1876); F. W. E. A. Kellner [Considerations on the theory of parallels as basis of geometry], Amsterdam, 1879, 81 pages; O. Simony, "Ueber eine Reihe neuer mathematischer Erfahrungssätze," *Sitzungsberichte der mathematisch-naturwissenschaftlichen Classe der Kaiserlichen Akademie der Wissenschaften*, Wien, volume 87 (1883), pages 556–587; volume 88 (1884), pages 939–974; F. M. Souvorof [Sur la formule générale pour la distance de deux éléments dans un système projectif à une dimension], *Proceedings of the Physico-Mathematical Section of the Naturalists' Society of the Imperial University of Kazan*,* volume I (1883) pages 9–11; J. C. Medeiros [Complément à la théorie des parallèles d'Euclide], *O Instituto, jornal scientifico e litteraris*, Coimbra, volume 30 (1883), pages 541–553; L. Kronecker [Clausius's coordinates in n-space], *Berliner Sitzungsberichte*, 1891, pages 881–890; Klein, "Vorlesungen über projektive Geometrie" (mimeographed), Cambridge, Mass., 1893;† A. Vasiliev, N. I. Lobachevsky, *El Progreso matematico*, volume 3 (1893), pages 137–139; J. N. Lyle, various papers in the *American Mathematical Monthly*, volumes 1–3 (1894–1896); in volume 3 (1896) of the *Monthly* there was also a paper by J. H. Drummond, in volume 11 (1904) one by E. B. Wilson, in volume 16 (1909) one by Wilczynski. No reference is given to Klein's "Elementarmathematik vom höheren Standpunkte aus," Teil II: Geometrie, Leipzig, 1909, or to an 8 page (large quarto size) sketch (in French) of Lobachevsky's life and works published, according to a pencilled note on my copy, at Kazan in 1886. The greater part (pages 581–593‡) of Note II ("Sur la géométrie non-euclidienne") in the seventh edition of Rouché and Comberousse's Géométrie, volume 2, Paris, 1900, should have been credited to Poincaré. "Rouché and Comberousse" is incorrectly listed under the date 1891, of its sixth edition; in the fifth edition of 1883 the notes treat of the same subjects as in the later editions. Miss Scott's articles in volume 1 (1900) of the *Mathematical Gazette* should surely be mentioned in connection with von Staudt's Geometrie der Lage, and if two English editions of Newcomb's Popular Astronomy are to be listed, reference should also be give to the 8 American, the 4 German, the Russian, and the Norwegian editions. The entry after "[Crelle, A. L.]" under 1835 should be put in the list for 1834. To the title under Eisenmenger's name on page 52 may be added: pp. 14+1 pl.

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* It is to be regretted that the names in connection with signed reviews are rarely given.

† Dr. Sommerville has, apparently, overlooked the Appendix of this volume.

‡ It is to be understood, of course, that not all the following titles would be noticed except in a work aiming to be as complete as the one under review.

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* Translation of Russian title.

† The edition of about 25 copies was published by some graduate students of Harvard University.

‡ E. Lebon in the Poincaré volume of the Savants du Jour series makes a slip in connection with this entry.

Incomplete as the work before us therefore appears, the mathematician must ever be grateful to Dr. Sommerville for this new, carefully prepared and admirably arranged aid to orientation in a vast field of mathematical science.

The volume is well printed by His Majesty's Stationers and the general get up (including the paper binding) is the same as for the International Catalogue.

R. C. ARCHIBALD.

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NOTES.

THE thirtieth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY will be held at the University of Chicago on April 5–6.

THE opening (January) number of volume 13 of the *Transactions of the American Mathematical Society* contains the following papers: "Ueber eine idealtheoretische Funktion," by E. LANDAU; "Theorems of oscillation for two linear differential equations of the second order with two parameters," by R. G. D. RICHARDSON; "The absolute minimum of a definite integral in a special field," by E. J. MILES; "An existence theorem for a problem of the calculus of variations in space," by E. G. BILL; "Linear algebras," by L. E. DICKSON; "A note concerning Veblen's axioms for geometry," by R. L. MOORE; "Natural families of curves in a general curved space of n dimensions," by JOSEPH LIPKE; "A class of periodic orbits of superior planets," by F. R. MOULTON; "Harmonic functions and Green's integral," by O. D. KELLOGG.

The opening (January) number of volume 34 of the *American Journal of Mathematics* contains: "Primitive groups with a determination of the primitive groups of degree 20," by Elizabeth R. Bennett; "On arithmetical theory of pencils of binary quadratic forms," by D. N. Lehmer; "On the principle of duality in the geometry of the sphere," by Arthur Ranum; "On the fundamental postulate of tamisage," by R. K. Morley; "Derivative domains of rationality," by S. Epsteen; "Critical revision of de Haan's Tables of definite integrals," by E. W. Sheldon. The frontispiece of the volume is a portrait of Maclaurin.

The December number (volume 13, number 2) of the *Annals of Mathematics* contains: "Points of indeterminate slope on the discriminant locus of an ordinary differential equation," by W. R. Longley; "Boundary problems and Green's functions for linear differential and difference equations," by Maxime Bôcher; "Conjugate directions on a hypersurface in a space of four dimensions and some allied curves," by C. L. E. Moore.

At the meeting of the London mathematical society held on December 14 the following papers were read: By J. W. Nicholson, "The pressure of radiation on a cylindrical obstacle"; by H. Hilton, "Hermitian invariants of a canonical substitution"; by E. W. Hobson, "The fundamental lemma of the calculus of variations and some related theorems"; by W. Burnside, "The outer isomorphisms of a group"; by E. B. Stouffer, "Invariants of linear differential equations"; by J. C. Fields, "A method of proving certain theorems relating to adjointness."

At the meeting of the Edinburgh Mathematical Society on January 12 the following papers were read: By J. A. Donaldson, "An infinite series of triangles and conics with a [illegible]ommon pole and polar"; by R. F. Davis, "The condition [illegible]hat the family of surfaces $\phi(x, y, z, \lambda) \doteq 0$ may be one of an orthogonal triad"; by F. E. Edwards, "Graphical trisection [illegible] circular arc."

The annual meeting of the Mathematical association was held in London, January 10. The following papers were read. Presidential address, by E. W. Hobson, "On the work of the international commission on mathematical teaching," by C.

Godfrey; "On some unrealized possibilities in mathematical education," by G. St. L. Carson; "A plea for the earlier introduction of the calculus, by C. V. Durell.

The United States Bureau of Education has recently issued Bulletins 13 and 16 for 1911, the former containing the report of the American Committees I and II, on mathematics in the elementary schools of the United States, the latter the report of Committees III and IV, on mathematics in the public and private secondary schools. These reports are prepared under the direction of the American Commissioners of the International commission on the teaching of mathematics. They may be secured gratis by addressing the United States Commissioner of Education at Washington.

At the University of Göttingen mathematical statistics has recently been added to the list of possible subjects from which the major and minors can be chosen for examination for the doctorate. Candidates in applied mathematics must henceforth be prepared to be examined in the mathematics of insurance.

Dr. G. Schirmer, a Chicago physician, has established at the University of Erlangen, a prize fund, to be known as the Helene Ottilie Schirmer foundation, in honor of his late wife. The income of about 600 marks is to be given to the author of the most meritorious thesis prepared at the university during a period of two years preceding each award; the subjects are to be in mathematics or physics in odd years, and in medicine in even years. The first award has recently been made to Dr. R. Baldus for his dissertation on certain line congruences.

At the Collège de France the chair of mathematics has been replaced by two, one in analysis and the other in geometry. Professor J. Hadamard has been appointed professor of analysis and Dr. M. d'Ocagne has been appointed professor of geometry. Dr. Vessiot has charge of the mathematical examinations for entrance and advancement. The lectureship under the Peccot foundation has been divided during the first half year it will be filled by Dr. E. Chazy and during the second half year by Dr. L. Chatenet. The Peccot prize for the year 1911 was awarded to Dr. M. Janet.

The Belgian academy has crowned a memoir on analysis by S. Bernstein and awarded the F. Deruyts prize for geometry to Dr. E. Fairon.

THE London mathematical society has awarded its De Morgan medal to Professor HORACE LAMB.

THE fiftieth scientific anniversary of Professor GASTON DARBOUX was celebrated at the Sorbonne on January 21.

PROFESSOR V. VOLTERRA, of the University of Rome, is giving a series of lectures at the University of Paris on the extension of the theory of functions, the integro-differential equations and integral equations, with applications. The first lecture was given January 22.

PROFESSOR A. VOSS, of the University of Munich, has been decorated with the Bavarian order of Maximilian.

PROFESSOR F. ENRIQUES, of the University of Bologna, has received the honorary degree of doctor of laws from the University of St. Andrews.

PROFESSOR J. J. THOMSON, of the University of Cambridge, has been elected foreign member of the royal academy of sciences of Naples.

PROFESSOR G. LAURICELLA, who had recently accepted a professorship of mathematics at the University of Rome, has at his own request resumed his former duties at the University of Catania.

PROFESSOR P. ANDOYER has been elected president of the Société mathématique du France.

DR. F. NUSL, of the Bohemian technical school at Prague, has been promoted to a full professorship of mathematics.

DR. R. BALDUS has been appointed docent in mathematics at the University of Erlangen.

DR. K. KNOPP has been appointed docent in mathematics at the University of Berlin.

PROFESSOR G. FABER, of the technical school at Stuttgart, has accepted a professorship of mathematics at the University of Königsberg.

DR. K. KOMMERELL has been appointed docent in mathematics at the technical school at Stuttgart.

DR. E. KRUPPA has been appointed docent in geometry at the University of Czernowitz.

PROFESSOR H. HENRIQUES, of the technical school at Stockholm, has been promoted to a full professorship of descriptive geometry.

PROFESSOR K. WIEGHARDT, of the technical school at Hanover, has accepted a professorship of mechanics at the technical school of Vienna.

DR. F. NOETHER has been appointed docent in mathematics at the technical school of Karlsruhe.

DR. H. STRÖLE has been appointed docent in mathematics at the University of Neuchâtel.

DR. A. TIMPE, of the technical school at Aachen, has been appointed docent in mechanics at the University of Münster.

PROFESSOR P. BURGATTI, of the University of Bologna, has been promoted to a full professorship of theoretical mechanics.

DR. Z. GIAMBELLI has been appointed associate professor of algebra at the University of Cagliari.

PROFESSOR E. C. COLPITTS, of the College of Emporia, Kansas, has been appointed professor of mathematics at the State College of Washington, at Pullman, Washington.

DR. A. B. FRIZELL, of the University of Kansas, has been appointed professor of mathematics in McPherson College, McPherson, Kansas.

THE death is announced of Professor GEORGE CHRYSTAL, of the University of Edinburgh, at the age of 60 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Anding (E.). Sechsstellige Tafeln der Besselschen Funktionen imaginaren Arguments. Leipzig, Engelmann, 1911. 8vo. 4+72 pp. Cloth. M. 8.80

Artusbrunnen (C.). Die Gleichung für 3 Potenzen gleicher Ordnung mit der Einschränkung des Fermat. Ein Beweis für die Grundregel der Mathematik. (Das Problem des Pythagoras.) 2te vermehrte und neudurchgesehene Ausgabe. Berlin, Buschhardt, 1911. 8vo. 4 pp. M. 2.00

Aussant-Carà (P.). Sulla discussione dei problemi riducibili al secondo grado. Con varie applicazioni. Livorno, Giusti, 1911. 8vo. 58 pp. L. 1.50

Bernadi (G.). Tavole contenenti i doppi i quadrati, i tripli dei quadrati ed i cubi dei numeri interi da 1 a 1000, ad uso degli istituti medi e superiori d'istruzione. 2a edizione rifatta. Bologna, Beltrami, 1911. 16mo. L. 1.50

Berzolari (L.). Geometria analitica, I: Il metodo delle coordinate. Milano, Hoepli, 1911. 16mo. 16+409 pp. L. 3.00

Böhm (F.). Über die Transformationen von homogenen bilinearen Differentialausdrücken. (Habilitationsschrift.) Munchen, 1911.

Dedekind (R.). Was sind und was sollen die Zahlen. 3te unveränderte Auflage. Braunschweig, Vieweg, 1911. 8vo. 21+58 pp. M. 1.80

Dingler (H.). Über die Bedeutung der Burali-Fortischen Antinomie für die Wohlordnungssätze der Mengenlehre. München, Ackermann, 1911.

Frechet (M.). See Heywood (H. B.).

Goldmann (H.). Der russische Rechenapparat, seine rationelle Anwendung in den Schulen und im praktischen Leben, nebst einigen Untersuchungen über den bildenden Wert der Arithmetik. Wien, Pichler, 1911. 8vo. 3+44 pp. M. 0.85

Heywood (H. B.) et Frechet (M.). L'équation de Fredholm et ses applications à la physique mathématique, avec une préface de M. J. Hadamard. Paris, Hermann, 1911. 8vo. 165 pp. Fr. 5.00

Jackson (D.). Über die Genauigkeit der Annäherung stetiger Funktionen durch ganze rationale Funktionen gegebenen Grades und trigonometrische Summen gegebener Ordnung. Göttingen, 1911 (Gekrönte Preisschrift.) 8vo. 98 pp.

Kiseljak (M.). Beiträge zur Theorie der vollkommenen Zahlen. (Croatian.) (Progr.) Agram, 1911.

Lalesco (T.). Introduction à la théorie des équations intégrales, avec une préface de E. Picard. Paris, Hermann, 1911. 8vo. 152 pp. Fr. 4.00

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THE WINTER MEETING OF THE CHICAGO SECTION.

THE twenty-ninth regular meeting of the Chicago Section of the American Mathematical Society was held at the University of Chicago on Friday and Saturday, December 29–30, 1911, extending through three half-day sessions. The total attendance was fifty-seven, including the following forty-three members of the Society:

Professor F. Anderegg, Professor W. H. Bates, Professor G. A. Bliss, Dr. H. T. Burgess, Professor H. E. Cobb, Professor D. R. Curtiss, Professor L. E. Dickson, Dr. Arnold Dresden, Professor Arnold Emch, Professor Peter Field, Dr. T. H. Hildebrandt, Dr. Louis Ingold, Professor Kurt Laves, Professor A. C. Lunn, Dr. W. D. MacMillan, Dr. H. F. MacNeish, Professor J. L. Markley, Professor William Marshall, Mr. R. M. Mathews, Professor Malcolm McNeil, Professor J. C. Morehead, Professor F. M. Morrison, Mr. E. J. Moulton, Professor F. R. Moulton, Professor Alexander Pell, Dr. Anna J. Pell, Professor W. S. Pemberton, Dr. R. E. Root, Mr. J. M. Rysgaard, Miss Ida M. Schottenfels, Mr. A. R. Schweitzer, Professor J. B. Shaw, Mr. T. M. Simpson, Professor C. H. Sisam, Professor H. E. Slaught, Professor A. W. Smith, Professor E. J. Townsend, Professor A. L. Underhill, Professor E. B. Van Vleck, Professor Marion B. White, Professor E. J. Wilczynski, Professor J. W. A. Young, Professor Alexander Ziwet.

Professor L. E. Dickson, Chairman of the Section, presided at the opening session on Friday morning. Professor E. B. Van Vleck presided at the session on Friday afternoon while Professor Dickson was delivering his address on the "History of the representations of numbers as the sum of squares," and Professor E. J. Townsend presided at the session on Saturday morning.

At the business meeting on Saturday morning the following officers of the Section for the year 1912 were elected: Professors D. R. Curtiss, chairman, H. E. Slaught, secretary, and A. L. Underhill, third member of the programme committee.

On Friday noon the members lunched together at the Quadrangle Club, and in the evening they dined together at the same place and spent one of the most enjoyable social occasions in the history of the Section.

The following papers were presented at this meeting:

(1) Professor ARNOLD EMCH: "Involutoric circular transformations as a particular case of the Steinerian transformation and their invariant net of cubics."

(2) Dr. R. E. ROOT: "Iterated limits in general analysis."

(3) Dr. ARNOLD DRESDEN: "Reduction of systems of linear differential equations of any order."

(4) Dr. LOUIS INGOLD: "Displacements in a function space."

(5) Professor L. E. DICKSON: "History of the representation of numbers as the sum of squares."

(6) Professor F. R. MOULTON: "Relations of families of periodic orbits in the restricted problem of three bodies."

(7) Professor L. E. DICKSON: "Note on Waring's theorem."

(8) Professor L. E. DICKSON: "Uniqueness of division in Cayley's algebras with eight units."

(9) Professor J. B. SHAW: "On differential invariants."

(10) Professor E. J. WILCZYNSKI: "On some geometric questions connected with the problem of three bodies."

(11) Professor PETER FIELD: "On Coulomb's laws of friction."

(12) Dr. E. G. BILL: "Analytic curves in non-euclidean space."

(13) Mr. H. F. VANDIVER: "Theory of finite algebras."

(14) Dr. ARNOLD DRESDEN: "Note on the second variation; Jacobi's equation and Jacobi's theorem in the calculus of variations."

(15) Professor G. A. MILLER: "Gauss's lemma and some related group theory."

(16) Professor R. D. CARMICHAEL: "On a class of linear functional equations."

(17) Professor R. D. CARMICHAEL: "On the theory of the gamma function."

Mr. Vandiver's paper was communicated to the Society through Professor Dickson. In the absence of the authors, the papers of Mr. Vandiver, Dr. Bill, Professor Miller, and Professor Carmichael were read by title. Abstracts of the papers, except the historical paper by Professor Dickson, follow below in the order of the list above.

1. It is well known that with every Steinerian, or involutoric quadratic transformation in a plane is associated an invariant net of cubics passing through the fundamental quadruple of

the transformation and its diagonal points. Conversely, every plane cubic admits of an infinite number of birational transformations into itself, which form a continuous group. The involutoric circular transformation in a complex plane

$$z' = \frac{az + b}{cz - a}$$

is a particular case of a Steinerian transformation, and it is the purpose of Professor Emch's paper to establish this equivalence and the principal properties of the two nets of cubics which remain invariant in the transformations of the first and second kind, respectively.

2. The paper by Dr. Root is essentially an amplification, together with some applications, of a method for the investigation of iterated limits outlined in his paper at the April meeting of the Society. The method involves a generalization of the idea of neighborhood of an element, in the sense that a neighborhood of an element is a subclass of the range class specially related to the element. This relation of subclass to element is taken as undefined, and as the basis of the system of postulates. The character and form of the postulates are determined largely by two fundamental requirements: first, to provide for an adequate treatment of ideal limiting elements, and second, to insure the persistence of the specified conditions under composition of systems. The theorems on functions are, for the most part, direct analogues or generalizations of well-known theorems on multiple sequences and multiple and iterated limits of functions of real variables. Certain special features pertain, however, to the presence of ideal elements. The general theory is found to be available for functions on any range, subject to the Frechet notion of voisinage, or to properly conditioned K_1 and K_2 relations of the type used by E. H. Moore.

3. The purpose of Dr. Dresden's paper is to derive conditions under which a system of n ordinary linear differential equations in $n + 2$ variables and of order p may be reduced to a system of the same type in $n + 1$ variables.

If the given equations are of the form

$$(1) \qquad \sum_{j=1}^{n+1} \sum_{l=0}^{p} a_{ilj} x_j^{(l)} = 0 \qquad (i = 1, \ldots, n),$$

where a_{ilj} are functions of t of class $C^{(l)}$ on a range (t_1t_2), we seek to determine conditions on the coefficients a_{ilj}, which will make it possible to write equations (1) in the form

$$\sum_{k=1}^{n}\sum_{l=0}^{p} A_{ilk} w_k^{(l)} = 0 \qquad (i = 1, \ldots, n),$$

where

$$w_k = \sum_{j=1}^{n+1} \alpha_{kj} x_j.$$

The conditions arrived at are expressible as the vanishing of determinants of order $n + 1$ in which the matrix a_{ipj} $(i = 1, \ldots, n; j = 1, \ldots, n + 1)$ forms the first n columns, the last column being formed by sums of derivatives of various orders of the coefficients a_{ilj}.

4. In the first part of Dr. Ingold's paper a set (φ) of three mutually orthogonal unit vectors φ_1, φ_2, φ_3, is considered in relation to any set (ψ) of unit vectors ψ_1, ψ_2, ψ_3 (not necessarily orthogonal) into which the first can be transformed linearly. The vectors of both sets are regarded as functions of any number of parameters $u_1, u_2, \ldots$.

The projections of the derivatives $\partial\phi_i/\partial u_l$ upon the vectors φ_j are denoted by $P_{ij}^{(l)}$; similarly the projections of the derivatives $\partial\psi_i/\partial u_l$ upon the vectors ψ_j are denoted by $Q_{ij}^{(l)}$. A set of fundamental relations (R) is obtained connecting the quantities $P_{ij}^{(l)}$ with the $Q_{ij}^{(l)}$. A special case is the set of equations of Darboux* satisfied by the two-parameter rotations of a rigid system.

If the sets of vectors (φ), (ψ) contain only two vectors each, the corresponding relations (R) still hold and in a special case reduce to a single equation for the two-parameter rotations of a rigid plane tangent to a given surface.

In the second part of the paper the corresponding developments are obtained for a set of n mutually orthogonal normed functions of x $(\alpha \leqq x \leqq \beta)$ and any number of parameters, $u_1, u_2, \ldots$, say $\varphi_1(x; u_1, u_2, \ldots)$, $\varphi_2(x; u_1, u_2, \ldots)$, $\ldots$, $\varphi_n(x; u_1, u_2, \ldots)$. Quantities analogous to the $P_{ij}^{(l)}$, $Q_{ij}^{(l)}$ are obtained and a set of relations analogous to the relations (R) is found connecting them. Special cases of this set are generalizations of

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* Darboux, Théorie générale des Surfaces, vol. 1, p. 49.

(1) The extensions to n dimensions and any number of parameters of Darboux's equations for the rotations of a rigid system.*
(2) Equations for the rotations of any rigid system moving tangent to any space.
(3) The Darboux extensions of the Lamé conditions for a triply orthogonal system of curves and surfaces in ordinary space.†

6. The theories of a number of classes of periodic solutions of the restricted problem of three bodies have been developed in papers presented to the Society by Professor Moulton in the last ten years. The present paper defines some additional periodic solutions consisting of orbits having multiple loops, examines the character of the changes through certain critical forms, and then takes up the problem of following out the analytic continuity, in the sense of analytic functions, of all the orbits with respect to the various parameters upon which the solutions depend, and of tracing the connections of the several families. Omitting reference to the nature of the branching and the definitions and qualifications necessary in some statements for complete precision, a few of the theorems on which the discussion depends are the following:

(1) Six orbits having real periods and single loops branch from one of zero dimensions at each of the finite masses, of which three are direct and three retrograde. In the vicinity of the origin only one each of the direct and of the retrograde orbits is real. They are symmetrical with respect to the x-axis.
(2) Six orbits having real periods and single loops branch from infinity, three of which are direct and three of which are retrograde with respect to fixed axes, while all are retrograde with respect to rotating axes. When the orbits are sufficiently large only one each of those which are direct and retrograde is real. They are symmetrical with respect to the x-axis.
(3) There is a geometrically single family of three-dimensional orbits branching from each of the five points of libration.

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* These extensions are given by Hatzidakis, "Displacements depending on one, two, ..., k parameters in a space of n dimensions." *American Journal*, vol. 22 (1900), pp. 154–184.

† Darboux, Leçons sur les Systèmes orthogonaux et les Coordonnées curvilignes, Book II, Chapter I.

Those branching from the collinear libration points are symmetrical with respect to the xy and xz-planes; those branching from the equilateral triangular points are symmetrical only with respect to the xy-plane unless the finite bodies are equal, when they are symmetrical also with respect to the yz-plane.

(4) There is a geometrically single two-dimensional family branching from each of the collinear libration points in which the motion is retrograde with respect to the points.

(5) There are two-dimensional orbits associated with the equilateral triangular libration points falling under a number of cases depending upon the ratio of the finite masses and the parameter employed.

(6) No two finite loops can unite at only a single point.

(7) Single loops develop when, and only when, an orbit passes through a form of ejection or acquires a cusp on a surface of zero relative velocity.

(8) Multiple loops develop when, and only when, an orbit has a corner at a collinear libration point.

(9) If a direct orbit passes through a form of ejection or a cusp form it acquires a loop; a retrograde orbit under the same circumstances loses a loop.

(10) Ejection forms and cusp forms of periodic orbits can disappear only by uniting with other similar forms.

(11) Isolated periodic orbits do not exist.

(12) Three direct orbits have infinite periods; no retrograde orbits have infinite periods.

7. This paper by Professor Dickson gives an algebraic proof of the existence of an identity expressing any elementary symmetric function of $x_1^2, \ldots, x_e^2$ of degree $2m$ in the x's as a linear function with rational coefficients of M powers, with exponent $2m$, of linear functions of $x_1, \ldots, x_e$ with integral coefficients, where M and all the coefficients depend upon m and e alone.

8. In the second paper by Professor Dickson, it is pointed out that the algebras given by Cayley (*Philosophical Magazine*, volume 26 (1845), page 210; *American Journal of Mathematics*, volume 4 (1881), page 293) are equivalent to that with the units $1, e_1, \ldots, e_7$, where

$$e_i^2 = -1, \quad e_ie_j = -e_je_i. \qquad (i, j = 1, \ldots, 7; \; i \neq j),$$

$$e_1e_2 = e_3, \quad e_1e_4 = e_5, \quad e_1e_6 = e_7, \quad e_2e_5 = e_7,$$

$$e_3e_4 = e_7, \quad e_3e_5 = e_6, \quad e_4e_2 = e_6,$$

with 14 equations obtained by permuting the subscripts cyclically. It is now shown that right and left hand division, except by zero, is always possible and unique in this algebra. This is done by expressing the general element as a linear function of e_2, e_4, e_6, with coefficients B, C, $\ldots$, linear in e_1, and applying

$$e_jB = \bar{B}e_j, \quad (Be_j)(Ce_j) = (B\bar{C})e_j^2, \quad (Be_j)(Ce_k) = (\bar{B}\bar{C})(e_je_k),$$

for $j, k = 2, 4, 6$; $j \neq k$. Here $\bar{B} = r - se_1$ if $B = r + se_1$.

9. This is the third paper by Professor Shaw along this line, the other two having been presented before the Chicago Section at the meetings in December, 1910, and April, 1911. The method previously presented is now generalized to the case of space of n dimensions. It is shown in a suitable notation that the curvatures of a region of $n - m$ dimensions are expressible as the scalar invariants of the operator which is the generalization of the quaternion $S\,()\,\nabla, \sigma$. The various differential parameters are treated in the same manner. Finally the relation to the Maschke symbols is pointed out.

10. Two years ago Professor Wilczynski showed in a paper read before the Chicago Section how certain geometric problems connected with the problem of three bodies might be formulated, and how a number of these led to the question whether isosceles solutions exist. In the present paper this question finds a complete answer. If no two of the three masses are equal, the only isosceles solutions are the well-known equilateral solutions of Lagrange and its limiting cases. If two of the masses are equal, there exist two essentially distinct types of isosceles solutions.

11. Coulomb's laws of friction state that when one body slides over another the frictional force is (1) proportional to the normal pressure, (2) independent of the relative velocity, (3) independent of the area of the surfaces which are in contact. Painlevé in his Leçons sur le Frottement has shown that if

we assume these laws, peculiar situations may arise in case the problem is of such a nature that the normal pressure is a function of the coefficient of friction. Professor Field in this paper treats by graphical methods a problem of this kind which has been treated analytically by Painlevé and others. The paper will be published in the *Zeitschrift für Mathematik und Physik.*

12. This is the first of a series of papers, which Dr. Bill purposes to present to the Society giving the results obtained by applying to non-euclidean geometry the methods of analysis used by Professor Eduard Study for euclidean space. Starting from certain fundamental identities existing between the absolute invariants of the group of motions, the author arrives, in the first part of his paper, at a complete classification of the analytic curves of non-euclidean space. The second part of the paper is taken up with the derivation of the Frenet-Serret formulas for regular curves.

13. The theory of finite fields, due to Moore (Mathematical Papers, Chicago Congress, 1893, pages 208–242), may be regarded as a generalization of the arithmetical theory of congruences with prime moduli. In the present paper Mr. Vandiver examines an algebra which is an extension of the finite field, and which is defined as follows:

A set of s distinct elements forms a finite algebra of order S if the elements can be combined by addition, subtraction and multiplication, these operations being subject to the commutative and associative laws of elementary algebra, and if the resulting sum, difference, and product be each uniquely determined as an element of the set. Also there must exist in the set at least one element u_k such that the equation $u_k x = u_0$ (u_0 being the zero element) has the unique solution $x = u_0$.

In this algebra division is not always possible, and if possible, may give more than one quotient. Some of the elementary properties of a general finite algebra are given. Elements are classified as units and non-units. Prime and composite non-units are defined. It would appear that the theory includes as particular phases all arithmetical congruence theories, in particular, when the congruences considered have composite moduli.

14. In Dr. Dresden's note, a method is given for the discussion

of Weierstrass's treatment of the second variation, Jacobi's equation, and Jacobi's theorem in the calculus of variations for the case of an integral of the form $\int F(x, y, x', y')dt$. The method is based on the reduction of differential forms and unifies these three subjects; it is, moreover, immediately extensible to cases where the integrand contains more unknown functions, as an illustration of which the space problem is taken up.

15. According to Gauss's lemma any number m which is not divisible by the prime number p is a quadratic residue or a quadratic non-residue of p according as the series m, $2m$, $\cdots$, $\frac{1}{2}(p-1)m$ includes an even or an odd number of numbers whose least absolute residues (mod p) are negative. The main objects of Professor Miller's paper are to exhibit the setting of this lemma in the theory of abelian groups and to show how readily it can be deduced from this theory.

A set of operators of a group G is called a complete set for the nth powers if the nth powers of these operators give all the different nth powers of the operators of G and if no two operators of the set have the same nth power. The products obtained by multiplying all the operators of a complete set for the nth powers of an abelian group by any operator of the group constitute a complete set for the nth powers. The h complete sets obtained by multiplying any one such complete set by the different operators of the abelian group whose orders divide n are called complementary sets. These complementary sets involve every operator of the abelian group once and only once. If all the numbers in any complete set for squares (mod p) of the numbers 1, 2, $\cdots$, $p-1$ are multiplied by any one α of these numbers, then α is a quadratic residue or a quadratic non-residue of p according as an even or an odd number of these products occur in the set which is complementary to the given set for squares. This theorem clearly includes Gauss's lemma.

16. It is a problem of great difficulty (and one which is not completely solved in any extensive class of cases) to determine the general solution of a functional equation which involves two operations, as in the case of partial differential equations or mixed equations. The object of the present paper is to investigate the properties of the solutions of an equation belonging

to this general class; namely, an equation of the form

$$f(x+\omega_1+\omega_2) + \varphi(x)f(x+\omega_1) + \psi(x)f(x+\omega_2) + \chi(x)f(x) = 0,$$

certain restrictions being placed on the coefficients $\varphi(x)$, $\psi(x)$, $\chi(x)$. Such an equation involves the two operations of replacing x by $x + \omega_1$ and by $x + \omega_2$. The general theory of these equations throws light on the nature of the solutions of various functional equations involving two operations.

Among the results which Professor Carmichael obtains are the following: The arbitrary elements in the solution of the equation are doubly periodic functions of periods ω_1 and ω_2. Regions in which the solutions may be chosen arbitrarily are found; and by means of such regions it is shown that an infinite number of arbitrary elements must enter into a general solution of the equation. Analytic solutions involving simply periodic functions (some of period ω_1 and others of period ω_2) are obtained. A means is also given for expressing linearly in terms of particular solutions of this type (the multipliers being doubly periodic functions) any one of a large class of quasi-general solutions.

17. From the general theory of difference equations (see papers by Carmichael and by Birkhoff in the *Transactions*, volume 12) it follows that the equation $g(x + 1) = xg(x)$ has two solutions $g_1(x)$ and $\bar{g}_1(x)$ having the property that each of the limits

$$\lim_{x=+\infty} g_1(x)x^{-x+\frac{1}{2}}e^{x} \quad \text{and} \quad \lim_{x=-\infty} x^{-x+\frac{1}{2}}e^{x}$$

exists and is 1, where x goes to infinity along the axis of reals in the positive sense in the first case and in the negative sense in the second case. Moreover,

$$\bar{g}_1(x) = (1 - e^{2\pi x\sqrt{-1}})g_1(x).$$

Also, any solution $g(x)$ is uniquely determined if $g(1)$ is assigned (different from zero) and

$$\lim_{x=+\infty} g(x)x^{-x+\frac{1}{2}}e^{x}$$

exists. It is natural to choose for the definition of $\Gamma(x)$ one that associates it most intimately with this general theory. Accordingly, Professor Carmichael defines a first and a second gamma function, $\Gamma(x)$ and $\bar{\Gamma}(x)$, as those solutions of the equation $g(x+1) = xg(x)$ which have the properties that

$$\Gamma(1) = 1, \qquad \bar{\Gamma}(x) = (1 - e^{2\pi x\sqrt{-1}})\Gamma(x),$$

and

$$\lim_{x=+\infty} \Gamma(x) x^{-x+\frac{1}{2}} e^{x} \quad \text{exists.}$$

If one starts from these definitions and makes use of the general theory of linear homogeneous difference equations of the first order, the fundamental properties of $\Gamma(x)$ and $\bar{\Gamma}(x)$ are readily obtained. The theory, as worked out recently by Professor Carmichael in his academic lectures, is decidedly simpler and more elegant than the usual theory of the gamma function, as developed, for instance, in Nielsen's Handbuch.

H. E. SLAUGHT,
Secretary of the Section.

AN IDENTICAL TRANSFORMATION OF THE ELLIPTIC ELEMENT IN THE WEIERSTRASS FORM.

BY PROFESSOR F. H. SAFFORD.

(Read before the American Mathematical Society, April 29, 1911.)

THIS paper is based upon a formula published in 1865 in a pamphlet entitled "Problemata quaedam mechanica functionum ellipticarum ope soluta.—Dissertatio inauguralis," by G. G. A. Biermann (Berolini), where it is quoted as derived from Weierstrass's lectures. The formula is, after correcting slight misprints in Biermann's pamphlet,

$$(1) \quad F(x) = x_0 + \frac{\sqrt{R(x_0)}\sqrt{S} + \frac{1}{2}R'(x_0)[s - \frac{1}{24}R''(x_0)] + \frac{1}{24}R(x_0)R'''(x_0)}{2[s - \frac{1}{24}R''(x_0)]^2 - \frac{1}{2}A \cdot R(x_0)}.$$

F is the solution of

$$(2) \quad (F')^2 = AF^4 + 4BF^3 + 6CF^2 + 4B'F + A' = R(F).$$

The accents used with F and R denote differentiation, x_0 is an arbitrary constant, and A, B, C, B', A' are constant coefficients. Also

$$
\begin{aligned}
&S = 4s^3 - g_2 s - g_3 = 4(s - \epsilon_1)(s - \epsilon_2)(s - \epsilon_3),\\
(3)\qquad &s = \wp(x), \quad g_2 = AA' + 3C^2 - 4BB',\\
&g_3 = ACA' + 2BCB' - AB'^2 - A'B^2 - C^3.
\end{aligned}
$$

In Enneper, Elliptische Functionen (Halle, 1890), pages 27–30, may be found considerable discussion of the preceding formulas, while on page 59 another form of (1) is used in obtaining the addition theorem for $\wp(u)$.

Greenhill, Elliptic Functions (London, 1892), page 151, mentions (1) in connection with the reduction of the general elliptic element $dx/\sqrt{X}$ to Weierstrass's canonical form $ds/\sqrt{S}$. Haentzschel's use of Weierstrass's formula will be considered later.

From (1) an identical transformation of $ds/\sqrt{S}$ may be obtained as follows: By a proper choice of constants α, β, γ, δ, the linear transformation

$$
(4)\qquad F = \frac{\alpha v + \beta}{\gamma v + \delta}
$$

changes (2) into

$$
(5)\qquad (v')^2 = 4v^3 - g_2 v - g_3,
$$

whence, writing

$$
(6)\qquad \bar{F} - x_0 = v,
$$

(5) becomes

$$
(7)\qquad (\bar{F}')^2 = 4(\bar{F} - x_0)^3 - g_2(\bar{F} - x_0) - g_3 = \bar{R}(\bar{F}).
$$

Hence

$$
\begin{aligned}
&\bar{R}(x_0) = -g_3, \quad \bar{R}'(x_0) = -g_2,\\
(8)\qquad &\bar{R}''(x_0) = 0, \quad \bar{R}'''(x_0) = 4! \quad \bar{R}''''(x_0) = 0.
\end{aligned}
$$

Then using the form $\bar{R}$ as indicated in (7) for R in the fundamental formula (1) gives

$$
(9)\qquad v = \frac{\sqrt{(-g_3)}\sqrt{(4s^3 - g_2 s - g_3)} - \frac{1}{2}g_2 s - g_3}{2s^2}.
$$

When (9) is rationalized and simplified by writing $g_2 = 4a$, $g_3 = 2b$, it gives

$$
(10)\qquad v^2 s^2 + 2avs + 2b(v + s) + a^2 = 0.
$$

Referring to (5), it appears that the general biquadratic X or $R(F)$ has now, by the changes above, become the same as in Weierstrass's elliptic element, so that the transformation (10) changes $ds/\sqrt{S}$ into itself, i. e., is an identical transformation, a fact which may be easily verified.

Equation (10) is obviously symmetrical in v and s, while (9) shows that the transformation is irrational in both directions. The curve corresponding to (10) has no node, except when $g_2^3 = 27g_3^2$, but consists of two distinct branches, asymptotic to both axes. Evidently the line $v - s = 0$ is an axis of symmetry and it may be shown to intersect the curve twice or not at all, according as $g_2^3 - 27g_3^2$ is negative or positive, this expression being also the criterion for the two types of $\wp$-function curves, the basis of Haentzschel's discussion.

Haentzschel, in his Reduction der Potentialgleichung (Berlin, 1893), wished to obtain a pair of families of orthogonal curves by the aid of conjugate functions. To this end he wrote

$$(11) \qquad s = \wp(t + iu), \quad \sigma = \wp(t - iu),$$

$$(12) \qquad x + iy = F(t + iu), \quad x - iy = F_1(t - iu),$$

in which F and its conjugate F_1 were defined by (2), and x_0 was complex. The orthogonal families came from the elimination of u or t respectively from (12) and writing either ρ or ρ_1 as a parameter, where

$$(13) \qquad \rho = \wp(2t) \quad \rho_1 = \wp(2iu).$$

He treated in detail the cases in which x_0 is any one of the roots of $R(x) = 0$, equation (2).

In the *Archiv der Mathematik und Physik,* series 3, volume 10, Hefte 3/4, pages 234–237, the writer has discussed some of Haentzschel's results and referred to other articles on the same point. On pages 33–37 (Potentialgleichung), Haentzschel treated the general case, i. e., x_0 unrestricted. This depends upon the $\wp$-function curves obtained from

$$(14) \qquad \begin{aligned} x + iy &= s = \wp(t + iu), \\ x - iy &= \sigma = \wp(t - iu) \end{aligned}$$

[ρ and ρ_1 as in (13), $a = g_2/4$, $b = g_3/2$], viz.,

$$(15) \qquad s^2\sigma^2 + 2\rho b + a^2 + (s^2 + \sigma^2)\rho^2 - s\sigma(s + \sigma)2\rho \\ + s\sigma(2a - 2\rho^2) + (s + \sigma)(2\rho a + 2b) = 0.$$

Haentzschel's work upon this topic concluded with an outline of the elimination of s and σ between (15) and two equations analogous to (10).

In the BULLETIN for June, 1899, pages 431–437, the writer has discussed the case in which

$$R(F) = A[\pm(F - x_0)^4 \pm c(F - x_0)^2 \pm 1], \tag{16}$$

and has factored the result, obtaining four curves each of the fourth degree.

The remainder of this paper is to be devoted to the case in which

$$\bar{R}(\bar{F}) = 4(\bar{F} - x_0)^3 - g_2(\bar{F} - x_0) - g_3. \tag{7}$$

First s and σ are to be eliminated from (15) and (10) together with the conjugate of the latter, i. e., from

$$v^2s^2 + 2avs + 2b(v + s) + a^2 = 0, \tag{17}$$

$$w^2\sigma^2 + 2aw\sigma + 2b(w + \sigma) + a^2 = 0, \tag{18}$$

$$[v = x + iy, \quad w = x - iy]. \tag{19}$$

In conclusion the resulting equation will be resolved into three components, of which two are the original ℘-function curves. Following Haentzschel, radicals may be avoided by writing (17) thus:

$$s^2 = -\frac{2s(av + b) + 2bv + a^2}{v^2}, \tag{20}$$

with a similar value for σ^2.
Substituting these values of s^2 and σ^2 in (15) gives

$$Ds\sigma + Ps + Q\sigma + E = 0, \tag{21}$$

$$\begin{aligned}
\tfrac{1}{2}D &= av^2w^2 - \rho^2v^2w^2 + 2\rho av^2w + 2\rho avw^2 + 2a^2vw + 2\rho bv^2 \\
&\quad + 2\rho bw^2 + 2abv + 2abw + 2b^2, \\
\tfrac{1}{2}P &= \rho av^2w^2 + bv^2w^2 + 2\rho bv^2w - \rho^2avw^2 + 2abvw + \rho a^2v^2 \\
&\quad - \rho^2bw^2 + a^3v + 2b^2w + a^2b, \\
\tfrac{1}{2}Q &= \rho av^2w^2 + bv^2w^2 + 2\rho bvw^2 - \rho^2av^2w + 2abvw + \rho a^2w \\
&\quad - \rho^2bv^2 + a^3w + 2b^2v + a^2b, \\
E &= 2\rho bv^2w^2 + a^2v^2w^2 - 2\rho^2bvw^2 - 2\rho^2bv^2w + 4b^2vw - \rho^2a^2v^2 \\
&\quad - \rho^2a^2w^2 + 2a^2bv + 2a^2bw + a^4.
\end{aligned} \tag{22}$$

Solving (21) for s and substituting in (17) gives a quadratic in σ, between which and (18), another quadratic, σ is to be eliminated. The result is

$$(23)\quad L^2 - b^2(PQ - DE)^2(4v^3 - 4av - 2b)(4w^3 - 4aw - 2b) = 0,$$

$$(24)\quad \begin{aligned} L = {} & v^2w^2E^2 - 2(PQ - DE)(av + b)(aw + b) \\ & + Pw^2[P(a^2 + 2bv) - 2E(av + b)] + Qv^2[Q(a^2 + 2bw) \\ & - 2E(aw + b)] + [D(a^2 + 2bv) - 2Q(av + b)][D(a^2 \\ & + 2bw) - 2P(aw + b)]. \end{aligned}$$

As would be expected, (21), (22), and (23) are symmetrical in v and w, while $(v - \epsilon_1)(v - \epsilon_2)(v - \epsilon_3)(w - \epsilon_1)(w - \epsilon_2)(w - \epsilon_3)$ from the last two factors of (23) may be used to find the foci of the curves represented. Each term in (24), on reduction, is found to contain the factor v^2w^2, and this being true of $PQ - DE$ which occurs also in (23), it is possible to cancel the factor v^4w^4 from the latter and at last obtain in v and w coordinates the equation of either of the pair of orthogonal curves. The factoring and consequent resolution of (23) is much facilitated by first treating the simple case for which $\rho = 0$. The degenerate form of (15) is

$$(25)\quad s^2\sigma^2 + 2as\sigma + 2b(s + \sigma) + a^2 = 0,$$

which is of the same type as (17) and (18). Elimination of s and σ from these three gives

$$(26)\quad [v^2w^2 + 2avw + 2b(v + w) + a^2]^3[\varphi(v, w)] = 0.$$

The last factor of (26) is the form which (15) becomes when $\rho = -4(2b^2 - a^3)/a^4$, $s = v$, $\sigma = w$. But the triple factor in (26) is not preserved when the general case $\rho \neq 0$ is treated. However (15), which it will be remembered is the equation of the $\wp$-function curves, has the remarkable property of being symmetrical in all *three* of the quantities ρ, s and σ.

By reasoning which will presently be given, it may be inferred that one factor of (23) is the first member of

$$(27)\quad \begin{aligned} & v^2w^2 + 2\rho b + a^2 + (v^2 + w^2)\rho^2 - vw(v + w)2\rho \\ & \qquad + vw(2a - 2\rho^2) + (v + w)(2\rho a + 2b) = 0. \end{aligned}$$

Now (23) is the result as stated of eliminating s and σ from the following:

$$
\begin{aligned}
(15)\quad & s^2\sigma^2 + 2\rho b + a^2 + (s^2 + \sigma^2)\rho^2 - s\sigma(s + \sigma)2\rho \\
& \qquad + s\sigma(2a - 2\rho^2) + (s + \rho)(2\rho a + 2b) = 0,
\end{aligned}
$$

$$
(17)\qquad v^2s^2 + 2avs + 2b(v + s) + a^2 = 0,
$$

$$
(18)\qquad w^2\sigma^2 + 2aw\sigma + 2b(w + \sigma) + a^2 = 0.
$$

Elimination of s from (15) and (17) gives an equation symmetrical in v and σ, which is thus identical with the result of eliminating w from (27) and (18), so that (27) is consistent with (15), (17) and (18). It may be shown by actual division that the *square* of (27) is a factor of (23). In carrying out the division, the symmetry of (27) as to ρ, v and w, in connection with the assumed symmetry of (23) in the same quantities, permits many terms of the quotient to be written by inspection after a few have been obtained. At this stage the knowledge of the factors for the special case $\rho = 0$ determines many coefficients in the general case. The entire computation has been carefully checked, and the single assumption of symmetry verified, but the length of the quotient (one hundred ninety-eight terms) prevents its reproduction in this paper. It is of particular importance that this identical transformation, when employed to obtain new curves from the ℘-function curves in connection with the theory of conjugate functions, reproduces the original curves.

UNIVERSITY OF PENNSYLVANIA,
April, 1911.

SURFACES IN HYPERSPACE WHICH HAVE A TANGENT LINE WITH THREE-POINT CONTACT PASSING THROUGH EACH POINT.

BY PROFESSOR C. L. E. MOORE.

(Read before the American Mathematical Society, December 27, 1911.)

THROUGH each point on a surface in ordinary space S_3 pass two tangents having with the surface three-point contact (tangents to the asymptotic lines). The osculating planes to these curves are also tangent to the surface at the point of osculation. It is easily seen that the lines on a ruled surface in hyperspace have these same properties. The question then naturally arises whether there exist in hyperspace surfaces not ruled having the same sort of lines. It is the object of this paper to show the existence of such surfaces and to derive the differential equation which the coordinates must satisfy. The results are generalized for spreads of higher dimensions than two.

1. The existence of such surfaces can be shown geometrically as follows: On a ruled surface in a space S_n trace three infinitely near curves c, c_1, c_2. Through the curves c_1 and c_2 pass a second ruled surface infinitely near the first and on this surface trace a third curve c_3 infinitely near to c_1, c_2. Then through the curves c_2, c_3 pass a third ruled surface infinitely near to the second and on it trace a curve c_4 infinitely near c_2, c_3. Continuing this process to the limit, it is at once evident that the locus of the curves c is a surface of the kind desired.

We will now derive the condition that a surface in S_n shall have through each point a curve whose osculating plane is tangent to the surface.

Let the surface be

$$x^{(i)} = x^{(i)}(u, v) \quad (i = 1, 2, 3, \cdots, n + 1) \tag{1}$$

and let three neighboring points be

$$(2)\quad x, \qquad (3)\quad x_1du + x_2dv,^*$$

$$x_{11}du^2 + 2x_{12}dudv + x_{22}dv^2 + x_1d^2u + x_2d^2v. \tag{4}$$

The tangent plane to the surface at the point x is defined by the three points[†]

$$x, \; x_1, \; x_2.$$

If the given curve is to have this plane for osculating plane, it is only necessary that the point

$$x_{11}du^2 + 2x_{12}dudv + x_{22}dv^2 \tag{5}$$

should lie in it; for the points (2), (3), and $x_1d^2u + x_2d^2v$ already lie in it. The condition then that (4) should lie in the tangent plane is

$$x_{11}du^2 + 2x_{12}dudv + x_{22}dv^2 + Ax_1 + Bx_2 + Cx = 0. \tag{6}$$

---

[*] Subscripts denote derivatives with respect to a variable t, where the curve on the surface is defined by assuming u and v to be functions of t. Hereafter superscripts will be omitted where no ambiguity occurs.

[†] See Segre, "Su una classa di superficie degl'iperspazii ecc.", *Atti di Torino*, 1907.

If the coordinates satisfy (6), then the points (2), (3), (4) will lie in the tangent plane. Hence:

If the coordinates of a surface in S_n satisfy a parabolic[*] *partial differential equation of the second order, then through each point of the surface passes a curve whose osculating plane coincides with the tangent plane to the surface at the point, and conversely.*

If now a tangent line is to have three-point contact with the surface, two conditions are necessary: (*a*) the point (5) must lie in the tangent plane; (*b*) for some choice of $d^2u : d^2v$ the point (4) must lie on the tangent line. The point (4) lies on the line joining (5) to $x_1d^2u + x_2d^2v$, and consequently if (6) is satisfied then when $d^2u : d^2v$ varies the point (4) will describe a line in the tangent plane. This line will cut the tangent line drawn in the direction du/dv. Thus if condition (*a*) is satisfied, condition (*b*) is also satisfied. Hence:

The necessary and sufficient condition that a surface in S_n have through each point a tangent line having three-point contact with the surface is that the coordinates of the surface satisfy a parabolic partial differential equation of the second order.

From the form of equation (6) it is seen that the directions of the three-point tangents are in the direction of the characteristics, since each is in the direction defined by du/dv.

In the paper referred to above Segre showed that a surface whose coordinates are particular solutions of two general partial differential equations of the second order must either be a developable or else lie in an ordinary space of three dimensions. From the above property it is evident that the surface could not be a developable if the coordinates satisfy two parabolic differential equations, for that would necessitate two principal directions (directions of three-point contact), which would be impossible. Then if such be the case, the surface must lie in an ordinary space S_3. In the pencil of partial differential equations which is defined by two general ones there are always two parabolic ones. Hence the two partial differential equations which the coordinates of a developable surface in hyperspace satisfy cannot be independent but must be such that there is only one parabolic equation belonging to the pencil defined by them.

2. The above considerations can be extended to varieties

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[*] See Segre, loc. cit., §15.

of any number of dimensions contained in a space S_n of n dimensions. Let the coordinates of the variety V_m be

$$x^{(i)} = x^{(i)}(u_1 u_2 \cdots u_m) \quad (i = 1, 2, \cdots, n+1). \tag{7}$$

The tangent S_m to V_m is defined by the points

$$x, \quad x_1, \quad x_2, \quad \cdots, \quad x_m$$

and if the osculating plane to a curve traced on the variety lies in this tangent S_m, then the three points

$$(8) \quad x, \qquad (9) \quad \sum_{i=1}^{i=m} x_i du_i,$$

$$\sum_{i=1}^{i=m} \sum_{j=1}^{j=m} x_{ij} du_i du_j + \sum_{i=1}^{m} x_i d^2u_i \tag{10}$$

must lie in it. But as before, (8), (9), and $\sum_{i=1}^{m} x_i du_i$ already lie in the tangent S_m and therefore the only condition necessary is that the point

$$\Sigma\Sigma x_{ij} du_i du_j \tag{11}$$

should lie in it. This condition is

$$\sum_{i,j=1}^{m} x_{ij} du_i du_j + \sum_{i=1}^{m} A_i x_i + A_{n+1} x = 0. \tag{12}$$

It is evident, since there are $m - 1$ independent quantities $du_1 : du_2 : du_3 : \cdots : du_m$ at our disposal and $m + 1$ quantities A, that equation (12) can always be satisfied if $2m > n$. And if $2m < n$, then the coordinates x must be particular solutions of (12) if the osculating plane to a curve traced on V_m lies in the tangent S_m. In this case, as in the preceding one, it is seen that if the point (11) lies in the tangent S_m, then the ratios $d^2u_1 : d^2u_2 : \cdots : d^2u_m$ can be so determined that (10) will lie in the line joining the points (8) and (9). Hence:

If through each point of a variety V_m in S_n there is a principal direction, then the coordinates must be particular solutions of the partial differential (12). *The osculating planes of curves enveloped by these principal directions will lie in the tangent S_m to the V_m.*

From equation (12) we see that the principal directions passing through each point of a hypersurface V_{n-1} in S_n form a quadric cone of $n - 2$ dimensions, and the principal directions through each point of a variety V_{n-k} $(2k<n)$ form a cone of order 2^k and dimensions $n-2k$. If $n=2m-1$ and $k=m-1$ the number of principal directions will be equal to 2^{m-1}. The curves enveloped by these directions correspond to the asymptotic curves on a surface in S_3 and in fact can be obtained by the consideration of conjugate directions just as in S_3. Take any two infinitely near points of V_m and draw the tangent spaces S_m to V_m at each of these points. The two S_m's will intersect in a line and this line is said to be conjugate to the direction defined by the two points of tangency. Using the same method as in a previous paper[*] by the author, the condition that the two directions du and δu be conjugate is found to be the vanishing of the $(m+2)$-rowed determinants of the matrix

$$\| x, x_1, x_2, \cdots, x_m, \sum_i \sum_j x_{ij} du_i \delta u_j \|.$$

The condition that the two conjugate directions should coincide is seen to be that the coordinates should satisfy equation (12). In this case then we have a new property of such curves, viz.: The tangent spaces S_m to V_m at two infinitely near points of the curve enveloped by the principal tangents intersect in the line joining the two points, that is, in a principal tangent. By an easy extension of the method used by Segre[†] it can be shown that this is a property of such curves in general. Or it may be seen that if this property does hold in general there is an S_{m+1} having two-point contact (i. e., tangent at two infinitely near points) with the variety V_m. Now the tangent at x is determined by

$$(13) \qquad x, \quad x_1, \quad x_2, \quad x_3, \quad \cdots, \quad x_m$$

and the tangent at the point

$$(14) \qquad \Sigma x_i du_i$$

is determined by

$$(15) \qquad \Sigma x_i du_i, \quad \Sigma x_{1j} du_j, \quad \Sigma x_{2j} du_j, \quad \cdots, \quad \Sigma x_{mj} du_j.$$

It is seen at once that the tangent space (13) contains the point (14) and that the tangent space (15) contains the point

$$(11) \qquad \Sigma x_{ij} du_i du_j$$

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[*] *Annals of Mathematics*, vol. 13, p. 89.

[†] Loc. cit.

Now if the space (13) contains (11), the two tangent spaces will have the line defined by the direction $du_1 : du_2 : \cdots : du_m$ in common. This is exactly condition (12). Hence:

The curves enveloped by the principal directions on a variety whose coordinates satisfy (12) *are such that the tangent S_m's at two consecutive points intersect in the line joining the two points.*

3. Let us now find the condition that a variety V_3 have through each point a tangent with four point contact with V_3. The condition may be got by first finding the condition that curves exist on V_3 whose osculating S_3 at any point coincides with the tangent S_3 to V_3 at that point. This requires that the two points

$$(16) \qquad \Sigma x_{ij} du_i du_j + \Sigma x_i d^2 u_i$$

$$(17) \qquad \Sigma x_{ijk} du_i du_j du_k + 3 \Sigma x_{ij} du_i d^2 u_j + \Sigma x_i d^3 u_i$$

should lie in the S_3 determined by the points

$$x, \quad x_1, \quad x_2, \quad x_3.$$

This is equivalent to the relations

$$(12) \qquad A \Sigma x_{ij} du_i du_j + \sum_1^3 B_i x_i du_i + Cx = 0,$$

$$(18) \quad A' (\Sigma x_{ijk} du_i du_j du_k + \Sigma x_{ij} du_i d^2 u_j) + \Sigma B_i' x_i du_i + C'x = 0.$$

In relation (12) the du_i define the direction of the curve. If (12) is satisfied, then the determinants of the matrix

$$\| \Sigma x_{ij} du_i du_j, \, x, \, x_1, \, x_2, \, x_3 \|$$

must vanish. Now if we change du_i into $du_i + \frac{1}{2} d^2 u_i$, this relation must still be satisfied; this leads to

$$\| \Sigma x_{ij} du_i d^2 u_j, \, x, \, x_1, \, x_2, \, x_3 \| = 0,$$

which used in (18) reduces that relation to

$$(19) \qquad A' \Sigma x_{ijk} du_i du_j du_k + \sum_1^3 B_i' x_i + C'x = 0.$$

Hence:

If a variety V_3 is covered with a family of curves whose osculating S_3 at each point coincides with the tangent S_3 to V_3 at that point, then the coordinates must be solutions of two simultaneous partial differential equations, one of parabolic type of *the second order and the other of the third order having coincident roots of its characteristic equation. The curves in question are the characteristics.*

If equation (12) is satisfied, we saw that the tangents in the direction du_i have three-point contact with V_3. If in addition (19) is also satisfied, since (17) lies in the tangent S_3, the quantities d^2u_i can be so chosen that (17) will lie on the line joining x to $x + dx$. Hence:

When the coordinates of a spread V_3 satisfy the differential equations (12) *and* (19), *through each point of the spread passes a tangent having four-point contact with the spread.*

In the same manner it can be shown that the necessary and sufficient condition that a spread V_m have in each point a tangent with r-point contact is that the coordinates should satisfy the $r - 2$ equations

$$(20)\quad \begin{aligned} &A\Sigma \frac{\partial^{r-1}x}{\partial u_i \cdots \partial u_{r-1}} du_1 du_2 \cdots du_{r-1} + \Sigma B_i x_i + Cx = 0, \\ &\cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot \\ &A^{(r-2)}\Sigma \frac{\partial^2 x}{\partial u_i \partial u_j} du_i du_j + \Sigma B_i^{(r-2)} x_i + c^{r-2}x = 0. \end{aligned}$$

The osculating spaces to such curves always lie in the tangent S_m to V_m. In particular when $r - 1 = m$ and the coordinates satisfy the system of equation (20), there is a curve passing through each point whose tangent lines have $(m + 1)$-point contact with V_m. The osculating S_m of the curve coincides with the tangent S_m to V_m.

That equations (20) be satisfied requires $(n - m)(r - 2)$ conditions. There are at our disposal $m - 1$ ratios $du_1 : du_2 : \cdots : du_m$; therefore if the above number is less than or equal to $m - 1$, the condition is always fulfilled. In case

$$(21)\qquad (n - m)(r - 2) = m - 1$$

the number of curves through each point is finite. When $n = 2k - 1$, $m = k$, $r = 3$ the number of three point tangents passing through each point is 2^k.

From (21) can be calculated the dimensions of V_m such that it have a finite number of tangents through each point having r-point contact. In particular if the spread is to have a 4-point tangent through each point, $m = \frac{1}{3}(2n + 1)$.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY.

NOTE ON MIXED LINEAR INTEGRAL EQUATIONS.

BY DR. W. A. HURWITZ.

(Read before the American Mathematical Society, September 12, 1911.)

By a mixed integral equation is here understood an equation in which an unknown function may be involved under several integral signs operating over regions of different numbers of dimensions, and in which also the values of the unknown function at isolated given points may occur. The highest number of dimensions of any region of integration may be termed the order of the equation; an equation entirely free from integral signs may also be regarded as a mixed integral equation of order zero. An equation in which regions of integration of different numbers of dimensions do not occur may be called pure; equations of this sort constitute the type usually studied.

The object of the present paper is to give some indication of the extent to which mixed (non-pure) *linear* integral equations may be expected to follow certain laws which have seemed to govern linear equations in general (e. g., pure integral equations and systems of algebraic equations). It is found that the analogy is not complete; the principal result, however, follows the lines of the usual theories. It is stated, for the case of equations of the first order, in the following

Theorem: *Let $f(x)$, $k_1(x)$, $k_2(x)$, $\cdots$, $k_m(x)$ be continuous fuctions of x in the interval*

$$R_1: \qquad a \leqq x \leqq b,$$

and $K(x, y)$ a continuous function of x and y in the region

$$R_2: \qquad a \leqq x \leqq b, \quad a \leqq y \leqq b;$$

let x_1, x_2, $\cdots$, x_m be distinct points of R_1. Then if the homogeneous mixed linear integral equation

$$(1) \qquad u(x) = \sum_{i=1}^{m} k_i(x)u(x_i) + \int_a^b K(x, s)u(s)ds$$

has no continuous solution in R_1, other than $u(x) = 0$, the non-homogeneous mixed linear integral equation

$$(2) \qquad u(x) = f(x) + \sum_{i=1}^{m} k_i(x)u(x_i) + \int_a^b K(x, s)u(s)ds$$

has one and only one continuous solution in R_1.

To prove this, let us form the two normal orthogonal sets of principal solutions for the kernel $K(x, y)$

$$\varphi_1(x),\ \varphi_2(x),\ \cdots,\ \varphi_n(x);\quad \psi_1(x),\ \psi_2(x),\ \cdots,\ \psi_n(x);$$

these satisfy the following conditions:

$$\varphi_j(x) = \int_a^b K(x, s)\varphi_j(s)ds;\quad \psi_j(x) = \int_a^b \psi_j(s)K(s, x)ds \qquad (j = 1, 2, \cdots, n);$$

$$\int_a^b [\varphi_j(x)]^2dx = \int_a^b [\psi_j(x)]^2dx = 1;\ \int_a^b \varphi_i(x)\varphi_j(x)dx = \int_a^b \psi_i(x)\psi_j(x)dx = 0 \quad (i \neq j;\ i, j = 1, 2, \cdots, n).$$

The equations

$$u_0(x) = f(x) - \sum_{j=1}^{n} \psi_j(x) \int_a^b \psi_j(s)f(s)ds + \int_a^b K(x, s)u_0(s)ds,$$

$$u_i(x) = k_i(x) - \sum_{j=1}^{n} \psi_j(x) \int_a^b \psi_j(s)k_i(s)ds + \int_a^b K(x, s)u_i(s)ds \qquad (i = 1, 2, \cdots m)$$

possess continuous solutions, since in each of them the known function outside the integral sign is orthogonal to $\psi_1(x)$, $\psi_2(x), \cdots, \psi_n(x)$. Form the following two systems of $m + n$ linear algebraic equations in $m + n$ unknowns $A_1, A_2, \cdots, A_m$; $B_1, B_2, \cdots, B_n$:

$$(\mathrm{I}_1)\quad A_p - \sum_{i=1}^{m} A_i u_i(x_p) - \sum_{j=1}^{n} B_j\varphi_j(x_p) = 0 \quad (p = 1, 2, \cdots, m),$$

$$(\mathrm{I}_2)\quad - \sum_{i=1}^{m} A_i \int_a^b \psi_q(s)k_i(s)ds = 0 \quad (q = 1, 2, \cdots, n),$$

$$(\mathrm{II}_1)\quad A_p - \sum_{i=1}^{m} A_i u_i(x_p) - \sum_{j=1}^{n} B_j\varphi_j(x_p) = u_0(x_p) \qquad (p = 1, 2, \cdots, m)$$

$$(\mathrm{II}_2)\quad - \sum_{i=1}^{m} A_i \int_a^b \psi_q(s)k_i(s)ds = \int_a^b \psi_q(s)f(s)ds \qquad (q = 1, 2, \cdots, n).$$

Either system (I) possesses a set of solutions $A_1, A_2, \cdots, A_m$; $B_1, B_2, \cdots, B_n$ not all zero, or else system (II) possesses a set of solutions.*

Under the second of these alternatives, the continuous function

$$u(x) = u_0(x) + \sum_{i=1}^{m} A_i u_i(x) + \sum_{j=1}^{n} B_j \varphi_j(x),$$

defined in terms of the set of solutions of (II), is a solution of (2), as may be verified by direct substitution in (2), with the aid of (II). A second solution of (2) is impossible if (1) has no solutions not identically zero, since the difference of two solutions of (2) is evidently a solution of (1).

Under the first alternative, the continuous function

$$u(x) = \sum_{i=1}^{m} A_i u_i(x) + \sum_{j=1}^{n} B_j \varphi_j(x),$$

defined in terms of the set of solutions (not all zero) of (I), is similarly a solution of (1). Finally, this solution $u(x)$ is not identically zero; for if $u(x) = 0$, then in particular $u(x_p) = 0$ $(p = 1, 2, \cdots, m)$, so that (by substituting the values x_p for x in the definition of $u(x)$ and making use of (I_1))

$$A_p = 0 \qquad (p = 1, 2, \cdots, m);$$

hence

$$0 = u(x) = \sum_{j=1}^{n} B_j \varphi_j(x),$$

and in virtue of the normal orthogonality of $\varphi_1(x)$, $\varphi_2(x)$, $\cdots$, $\varphi_n(x)$,

$$B_j = 0 \qquad (j = 1, 2, \cdots, n),$$

in contradiction to the fact that some one of $A_1, A_2, \cdots, A_m$; $B_1, B_2, \cdots, B_n$ is not zero.

The theorem stated above has thus been justified. With this theorem, however, the exactness of the analogy to the ordinary theories of linear equations seems to end. The more detailed study of the mixed linear integral equation is reserved for a later communication; an indication of the differences which present themselves is furnished by the following remark, which is stated without proof. The rôle of the adjoint homogeneous equation in the ordinary theories* is here taken by a pure homogeneous linear integral equation together with linear integral auxiliary conditions

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* The fact that these two states of affairs are not mutually exclusive is unessential.

$$(1') \qquad v(x) = \int_a^b v(s)K(s, x)ds; \quad \int_a^b v(x)k_i(x)dx = 0 \qquad (i = 1, 2, \cdots, m);$$

but the numbers of independent solutions of (1) and (1′) need not, as in the usual cases, be the same. For example, there can never be more (and will in general be fewer) than n linearly independent continuous solutions of (1′), while there may under certain conditions be as many as $m + n$ linearly independent continuous solutions of (1).

CORNELL UNIVERSITY,
December 22, 1911.

NOTE ON THE GRAPHICAL SOLUTIONS OF THE FUNDAMENTAL EQUATIONS IN THE SHORT METHODS OF DETERMINING ORBITS.

BY PROFESSOR A. O. LEUSCHNER AND MR. B. A. BERNSTEIN.

(Read before the San Francisco Section of the American Mathematical Society, October 28, 1911.)

THE chief difficulty in the determination of preliminary orbits of new comets or asteroids arises from the fact that the geocentric or heliocentric distance of the new body enters into the problem as an unknown quantity. In the older methods the heliocentric distance often is assumed to be unity as a first approximation in the determination of comet orbits. The average heliocentric distance of the known asteroids is generally chosen as a first approximation for new asteroids. The true distances are then evaluated by laborious approximations.

In the short methods proposed by Professor Leuschner it is possible to make a direct determination of the geocentric distance at the middle date of the three underlying observations to as many decimals as may seem desirable. This is accomplished by a graphical solution in the determination of parabolic or circular orbits and by means of a table of geocentric distances for general orbits.

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* E. g., for the pure linear integral equation, the equation with transposed kernel; for the system of linear algebraic equations, the system with transposed (or conjugate) matrix.

For the parabola or circle the fundamental equation is of the form

$$f(z) = (z - p')^2 + q'^2 - \frac{h}{[(z - c)^2 + s^2]^{\frac{1}{2}}} = 0,$$

where the unknown z represents the ratio of the geocentric distance of the body to the geocentric distance of the sun and where all other quantities are known constants. The roots of the equation are given by the intersections of the parabola

$$y = (z - p')^2$$

with the asymptotic curve

$$y = \frac{h}{[(z - c)^2 + s^2]^{\frac{1}{2}}} - q'^2.$$

The distance between the axes of symmetry of the two curves is $c - p'$, reckoned from the axis of the parabola. The equations may be written

$$y = z'^2, \quad \eta = \frac{h}{(\zeta^2 + s^2)^{\frac{1}{2}}},$$

where $z' = z - p'$, $\zeta = z - c$, $\eta = y + q'^2$. The equation of the asymptotic curve may also be written

$$\zeta = s \tan \theta, \quad \eta = \frac{h}{s} \cos \theta.$$

In the astronomical problem only the positive real roots of $f(z) = 0$ need to be considered. There are either one or three positive real roots. The portions of the asymptotic curve near its intersections with the parabola may be conveniently plotted (Fig. 1) without the aid of a trigonometrical table, as follows: Draw a circle with radius h/s about the origin $\eta = 0$, $\zeta = 0$ as center and draw the line $\eta = s$. Draw any vector from the origin making an angle θ with $\zeta = 0$. Through the intersection M of vector and circle draw MM'

parallel to $\eta = 0$ and through the intersection N of vector and $\eta = s$ draw NN' parallel to $\zeta = 0$. Then for the intersection P of MM' and NN' we have

$$\zeta = s \tan \theta \quad \text{and} \quad \eta = \frac{h}{s} \cos \theta$$

and, therefore, P is a point on the asymptotic curve.

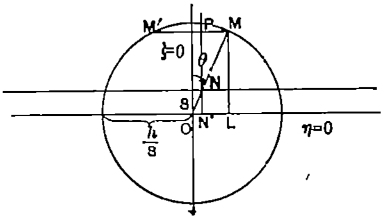

FIG. 1.

This construction was independently suggested by Mr. B. A. Bernstein. In practice the parabola $y = z'^2$ is first placed in proper position (with its vertex at $\zeta = p' - c$, $\eta = q'^2$ and its axis parallel to $\zeta = 0$) and a point P is located on either side of an intersection. The values of ζ corresponding to the intersections may then be read off and finally the corresponding values of z are given by $z = \zeta + c$. In locating a point P it is not necessary actually to draw either the vector or MM' or NN', but P may be located by means of rulers held in the proper position. In case of more than one solution the corresponding roots z may thus be obtained at one and the same time.

In the solution of an orbit without hypothesis regarding the eccentricity the equation is of the form

$$(z^2 - 2c + 1)^3 (z - m)^2 - m^2 = 0.$$

There may be either one or two positive real roots which may be taken from a table with ψ and $1/m$ as arguments where $\psi = \cos^{-1} c$ and m are known constants. The table is an extension of Table XIII*a* of von Oppolzer, Lehrbuch der Bahnbestimmung, volume 1.

In case of three parabolic solutions, the physical solution is determined by identification with a general solution. The mathematical parabolic solutions may thus be rejected in the course of computation. The only known cases with three parabolic solutions are comet Cruls of 1882 and comet 1910*a*. Graphical solutions for these comets have been made by Mr. E. S. Haynes, John H. Pitman, and Miss S. Levy.

An application of the foregoing methods of solution for comet 1910*a* has been published by Professor Leuschner in the *Lick Observatory Bulletin*, No. 197. A full account with the tables is being printed in the *Publications of the Lick Observatory*, volume 7.

Mr. Bernstein gives a purely geometric solution of the fundamental equation that enters into the *general* solution of an orbit. He proceeds as follows:

From the equation

$$(1)\qquad (z^2 - 2c + 1)^3(z - m)^2 - m^2 = 0$$

it follows that

$$z - m = \frac{m}{(z^2 - 2c + 1)^{\frac{3}{2}}},$$

where only the positive sign before the radical need be considered.

Making the substitution $x = z - c$, and writing $1 - c^2 = s^2$ (since $c < 1$), the above becomes

$$(2)\qquad x + c - m = \frac{m}{(x^2 + s^2)^{\frac{3}{2}}}.$$

The roots of this equation are got from the intersections of the curve

$$(3)\qquad y = \frac{1}{(x^2 + s^2)^{\frac{3}{2}}}$$

with the straight line

$$(4)\qquad my = x + c - m.$$

If we make the substitution $x = s \cot \theta$, we get

$$(5)\qquad \begin{aligned} x &= s \cot \theta, \\ y &= \frac{1}{s^3} \sin^3 \theta = \frac{3}{4s^3} \sin \theta - \frac{1}{4s^3} \sin 3\theta \end{aligned}$$

as the parametric equations of the curve represented by (3). This curve can now be constructed geometrically (Fig. 2) as follows:

Draw a line $\overline{A'A}$ parallel to the x-axis at a distance s above it. With the origin O as center construct two circles having for radii $1/4s^3$ and $3/4s^3$ respectively. Let θ be any angle which a line $\overline{OQ}$ makes with $\overline{OX}$, the positive direction of the x-axis, and let $\overline{OQ}$ meet $\overline{A'A}$ in R and the circle of radius $3/4s^3$ in Q. Draw the angle $XOT = 3\theta$, and let the terminal side of this angle meet the circle of radius $1/4s^3$ in T. Draw $\overline{RM}$ perpendicular to $\overline{OX}$. From T and Q drop perpendiculars to $\overline{RM}$, meeting the latter in K and L respectively. On $\overline{MR}$ lay off $\overline{MP} = \overline{KL}$. Then P is a point on the required curve.

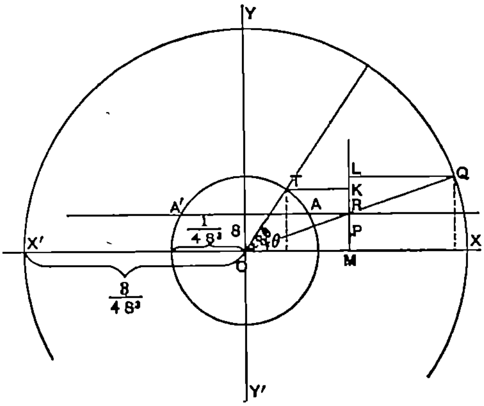

FIG. 2.

For

$$x = \overline{OM} = s \cot \theta$$

and

$$y = \overline{MP} = \overline{KL} = \overline{ML} - \overline{MK} = \frac{3}{4s^3} \sin \theta - \frac{1}{4s^3} \sin 3\theta$$

We thus obtain the geometric construction of the curve represented by equation (3). Its intersections with the line given by (4) are then easily found, and hence equation (2), and therefore (1), are solved.

ON A FUNCTIONAL EQUATION.

BY MR. A. R. SCHWEITZER.

(Read before the American Mathematical Society, October 28, 1911.)

1. ABEL,[*] in a discussion rigorized by Stäckel,[†] has shown (in effect) that if

$$f(x, u) = f(y, v) = f(z, w), \tag{1}$$

where

$$f(y, z) = u, \quad f(x, z) = v, \quad f(x, y) = w, \tag{2}$$

then there exists a function χ, dependent upon the function f, such that

$$\chi(f(x, y)) = \chi(x) + \chi(y).$$

In this paper it is desired to establish the following theorem: *If* $f(v, w) = u$,[‡] *then there exists a function* χ *such that*

$$\chi(f(x, y)) = \chi(x) - \chi(y).$$

In a sense, this theorem is a correlative of the theorem of Abel.

2. The method of proof of the preceding theorem resembles the procedure of Abel and Stäckel, l.c. Differentiating the relation

$$f(v, w) = u \tag{3}$$

with respect to y, z, x, we have

$$0 + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial y} - \frac{\partial u}{\partial y} = 0,$$

$$\frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial z} + \quad 0 \quad - \frac{\partial u}{\partial z} = 0,$$

$$\frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial x} - \; 0 = 0.$$

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[*] *Crelle's Journal*, vol. 1 (1826), p. 11; cf. M. Cantor, *Zeitschrift für Mathematik und Physik*, vol. 41 (1896), p. 161.

[†] *Zeitschrift fur Mathematik und Physik*, vol. 42 (1897), p. 323; cf. P. Hayashi, same journal, vol. 44 (1899), p. 346.

[‡] This relation, with the preceding definitions of u, v, w, we call quasi-transitiveness. Abel, l.c., bases his discussion on the symmetry of $f(z, w)$ in x, y, z.

Therefore, on elimination of $\partial f/\partial v$, $\partial f/\partial w$,

$$\begin{vmatrix} 0 & \partial w/\partial y & \partial u/\partial y \\ \partial v/\partial z & 0 & \partial u/\partial z \\ \partial v/\partial x & \partial w/\partial x & 0 \end{vmatrix} = 0$$

or

$$(4) \qquad \frac{\partial v}{\partial x}\cdot\frac{\partial w}{\partial y}\cdot\frac{\partial u}{\partial z} = -\frac{\partial v}{\partial z}\cdot\frac{\partial w}{\partial x}\cdot\frac{\partial u}{\partial y}.$$

The relation (4) is, of course, much more general than relation (3) from which it is derived, since (4) can be obtained, for instance, by differentiating any one of the relations*

$$\phi(v, w) = u, \quad \phi(w, u) = v, \quad \phi(v, u) = w.$$

Relation (4) differs, moreover, only in sign from a relation derived from (1) by Abel and Stäckel, and hence proceeding in the manner of the latter, we obtain from (4), if z = constant,

$$(4') \qquad \frac{\partial w}{\partial y}\cdot\psi'(x) = -\frac{\partial w}{\partial x}\cdot\psi'(y)$$

and as a solution† of (4′),

$$f(x, y) = \Omega(\psi(x) - \psi(y)),$$

where Ω remains to be further conditioned. Substituting in (3), we get

$$(5) \qquad \Omega[\psi\Omega(\xi - \zeta) - \psi\Omega(\xi - \eta)] = \Omega(\eta - \zeta),$$

where

$$\psi(x) = \xi, \quad \psi(y) = \eta, \quad \psi(z) = \zeta.$$

On differentiating (5) with respect to y and z, and comparing the resulting relations, we find

$$(6) \qquad \frac{\partial}{\partial\eta}\psi\Omega(\xi - \eta) = \frac{\partial}{\partial\zeta}\psi\Omega(\xi - \zeta).$$

Now we differentiate (6) with respect to y; then since

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* It may perhaps be remarked in passing that $f(v; w) = u$ follows from $\phi(w, u) = v$ and $f(\phi(x, y), x) = y$.

† Cf. Abel, *Crelle's Journal*, vol. 2 (1827), p. 389. See also Mansion-Maser, Partielle Differentialgleichungen, Berlin (1892), pp. 34–37.

$$\frac{\partial}{\partial \eta}\psi\Omega(\xi-\eta) = -\frac{\partial}{\partial(\xi-\eta)}\psi\Omega(\xi-\eta)$$

we have

$$\frac{\partial^2}{\partial(\xi-\eta)^2}\psi\Omega(\xi-\eta)\cdot\frac{\partial\eta}{\partial y} = 0.$$

If $\partial\eta/\partial y = 0$, then $\psi(y) = \text{constant}$, and hence $f(x, y) = \text{constant}$, which we exclude; therefore

$$\frac{\partial^2}{\partial(\xi-\eta)^2}\psi\Omega(\xi-\eta) = 0$$

or

(7) $$\psi\Omega(p) = cp + c_1, \quad p = \xi - \eta,$$

where c and c_1 are constants. Now let

(8) $$\chi(x) = \psi(x) - c_1;$$

then

$$\chi(\Omega p) = cp,$$

and consequently,

(9) $$\chi(f(x, y)) = c(\chi(x) - \chi(y)).$$

The constant c can be determined by substituting in (9) in accordance with (3); we find $c^2 - c = 0$, that is, since $f(xy) \neq \text{constant}$, $c = 1$. Hence the theorem under § 1 is verified. Conversely, if

(10) $$f(x, y) = \chi^{-1}(\chi(x) - \chi(y)),$$

then $f(x, y)$ has the property (3). If $f(x, y)$ is assigned, then by following Abel, l. c., volume 1, page 14, the function χ can be determined by the formula

(11) $$\chi(x) = -\chi'(y)\cdot\int\frac{\partial w/\partial x}{\partial w/\partial y}dx,$$

where y is to be regarded as a constant. From this determination we deduce

1°. If $f(x, y) = \phi(x/y)$, then

$$\log c\cdot\phi(x/y) = \log cx - \log cy$$

and therefore, $\phi(x/y) = c'\cdot x/y$.

2°. If $f(x, y) = \phi(x-y)$, then $\phi(x-y) = x - y + c''$, where c' and c'' are constants.

In the case of the equation

$$(12) \qquad f(v, u) = w$$

a discussion similar to the above is valid. It may be observed that relation (12) follows from (3) and (10), as does also a part of relation (1), viz.,

$$f(y, v) = f(z, w).$$

3. We conclude with a few remarks concerning the nature of the generalization of relation (3), which is perhaps less obvious than the corresponding generalization of relation (1) of Abel. Presumably the relation (3) occurs in a sequence of functional relations

$$f(f(t_1, x_2), f(t_1, x_1)) = f(x_1, x_2),$$

$$f(f(t_1, t_2, x_3), f(t_1, t_2, x_2), f(t_1, t_2, x_1)) = f(x_1, x_2, x_3),$$

$$f(f(t_1, t_2, t_3, x_4), f(t_1, t_2, t_3, x_3), f(t_1, t_2, t_3, x_2), f(t_1, t_2, t_3, x_1)) = f(x_1, x_2, x_3, x_4),$$

etc. This sequence leads to the sequence of linear partial differential equations of the first order

$$\frac{\partial u_2^{(0)}/\partial x_2}{\psi_2'(x_2)} + \frac{\partial u_2^{(0)}/\partial x_1}{\psi_2'(x_1)} = 0,$$

$$\frac{\partial u_3^{(0)}/\partial x_3}{\psi_3'(x_3)} + \frac{\partial u_3^{(0)}/\partial x_2}{\psi_3'(x_2)} + \frac{\partial u_3^{(0)}/\partial x_1}{\psi_3'(x_1)} = 0,$$

$$\frac{\partial u_4^{(0)}/\partial x_4}{\psi_4'(x_4)} + \frac{\partial u_4^{(0)}/\partial x_3}{\psi_4'(x_3)} + \frac{\partial u_4^{(0)}/\partial x_2}{\psi_4'(x_2)} + \frac{\partial u_4^{(0)}/\partial x_1}{\psi_4'(x_1)} = 0,$$

etc., where

$$f(x_1, x_2) = u_2^{(0)}, \quad f(x_1, x_2, x_3) = u_3^{(0)}, \quad f(x_1, x_2, x_3, x_4) = u_4^{(0)},$$

etc. The preceding equations have respectively the solutions,*

$$u_2^{(0)} = F_2(\psi_2(x_1) - \psi_2(x_2)),$$

$$u_3^{(0)} = F_3(\psi_3(x_1) - \psi_3(x_2), \quad \psi_3(x_1) - \psi_3(x_3)),$$

$$u_4^{(0)} = F_4(\psi_4(x_1) - \psi_4(x_2), \quad \psi_4(x_1) - \psi_4(x_3), \quad \psi_4(x_1) - \psi_4(x_4)),$$

etc., where F_2, F_3, F_4, etc., are arbitrary functions to be conditioned by the original functional relations.

CHICAGO, ILL.,
July, 1911.

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* Mansion-Maser, l.c.

SHOP MATHEMATICS.

In the December issue of the Bulletin there appeared a criticism of Holton's Shop Mathematics by Professor Charles N. Haskins which was in some respects so manifestly unjust as hardly to merit a reply. But since the criticism appears in a journal for which we have always had the highest regard, and since moreover there is an implied attack upon the ideals of secondary technical education, it may be well to show wherein this criticism falls wide of its intended mark.

The fundamental error under which Professor Haskins labors is the assumption, entirely unwarranted, that this book indicates the character of the mathematical instruction in technical high schools. As a matter of fact it was prepared mainly for the use of mechanics in the shop mathematics classes of an evening trade school, and finds simply an incidental, though very valuable, use in the (day) Technical High School, which is a school of full academic training, preparing students not only for industrial work directly but also for further study in the highest technical schools and colleges in the country. To meet the admission requirements of these institutions it is obviously necessary to give full courses in mathematics. And entirely aside from the preparatory courses no sensible teacher would fail to recognize the value of the purely mathematical element as a basis for application in the industries, as Professor Haskins points out at the close of his paper. This is not only assumed, but it is emphasized in all strong technical courses. The unfairness, or perhaps I should say the inaccuracy, of Professor Haskins's charge appears plainly in a sentence on page 138 in which he quotes, or misquotes, from the introduction a sentence which as there printed reads, "No attempt is made in this book to teach mathematical theory or principles." That is true. No such attempt is made *in this book;* for the purpose of the book does not require it. But Professor Haskins leaves out the phrase "in this book" and suggests that the author of the introduction was referring to the "character of the mathematical instruction in such schools." Professor Haskins therefore apparently misunderstood not only the kind of school for which the book was mainly intended,—viz., a trade school—but also the use which is actually made of it in the Technical High School—mistakes which really take away all the point in his attempt at serious criticism.

And yet there seems to be an element of unfairness in the general spirit of the criticism which should not go unchallenged. Professor Haskins assumes so much ignorance of practical affairs on the part of the author of this book, who is really a very practical man, and gives such a ridiculous turn to some of the problems selected for criticism as to provoke amazement, if not laughter, at the expense of the critic himself. For example, note the remarks about problem 17 on the raising of a building by means of jack screws. Now that was a problem based on facts. It was involved in the moving of one of our large public buildings within a stone's throw of the Springfield Technical High School. There were actually one hundred ¼ inch lead jack screws used; but it is by no means necessary to assume that there were one hundred men concerned in this operation,—one stationed at each jack screw. As a matter of fact there were five men, one for each of the five rows of jack screws, who operated each of the twenty jack screws in his row in order, giving a short turn at a time to each; and thus the building, which was a brick building, was successfully raised and moved. There is really no occasion for ridicule,—for comparing the force used to that exerted by bending the "little finger," or for comparing the bar with a "pen holder"; and it is inconceivable to the writer how anyone could find in this problem the necessary conception of one hundred laborers employed. And yet this ridiculous conception seems to have been suggested to Professor Haskins's mind by this by no means unusual case of lifting a brick building.

Turning now to the computations involved in this problem, it would hardly seem to be necessary to point out that this was not really given as a problem of computation that would occur in actual practice. It was expressly stated at the beginning that the element of friction was left out. It was therefore not intended to be a complete and actual problem at all. In fact, no contractor would need to figure out the length of these jack screw bars or the force required. For him and for most workmen such a matter is one of judgment acquired by experience. The problem was obviously chosen to illustrate the underlying *operating principles* of jack screws as *expressed in the formula;* and for this purpose there is certainly no harm in getting a computed result to thousandths of an inch. It is convenient to have such answers in order to test up the accuracy of the arithmetical work of men in trade schools; for they need practice in computation quite as much as they need anything. But there is hardly one of these men who would not laugh down the critic who assumes that the author, or even they themselves, would be so lacking in common sense as to consider such a computed result as finding actual application in the practice of raising buildings. The interest of this problem for these men obviously consists in the insight it gives into the meaning of a certain mechanical formula. It is for the engineer, or for the workman who aspires to become an engineer, to make corrections for friction and figure out, if necessary, the value of the force that could actually be applied with a jack screw of known dimensions and determine the number required. But that is an entirely different matter.

The same reply may be given in general to the second criticism upon problem 12, page 51. Here again the main purpose is to illustrate the combination of several formulas; and the calculation is carried to its full extent in order that the work of the several members of the class may be checked up for accuracy. Friction is disregarded, the effort being to illustrate the mathematical basis on which practical work depends. Practical work itself would obviously involve many corrections for friction and other losses and imply approximate results.

Again in the criticism of the problem on spiral gears (page 45) Professor Haskins finds in this a good example of how *not* to apply mathematical formulas in the solution of such problems because, as he says, the method of solution given involves first multiplying by π and then dividing by π. Obviously this is unnecessary in a mathematical computation *to find one result*,—as, for example, the number of teeth. As a matter of fact, however, in this problem four results have to be found, viz., the circular pitch, the number of teeth, the blank diameter, and the lead of the spiral, all four of which results must be known in order to set up the machine to cut the gears. This is plainly indicated in the problem as given in the book under criticism in the words, "Calculate the other dimensions of the gears." Professor Haskins chooses to state the problem as one to determine the number of teeth (N) only,—a gross misstatement which, it would seem, could hardly have been made for any other purpose than to manufacture a point against the book. And yet we wish to be charitable. One accustomed to think along the lines of pure mathematics only would not, of course, appreciate the real meaning of the mathematical layout of this problem; and it is to be presumed that this is Professor Haskins's real difficulty. He deprecates unnecessary mathematical work; and so do we. Short cuts in arithmetical computations are of course desirable if the workmen can be safely trusted to carry them out; but practical teachers find that it is not wise to dwell much upon short cuts in the elementary stages of the work. Since most of the men who use this book are far from being experts in computation, short cuts would be on the whole confusing, though of course when they are not confusing they are to be recommended. The kind of formalism Professor Haskins criticizes is not "excessive" and "pernicious" so long as it is necessary for a thorough understanding of the processes of computation.

CHARLES F. WARNER.

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The above statements may be answered as follows:

(1) The review is said to contain "an implied attack on the ideals of secondary technical education." On the contrary those ideals, as stated by Dr. Woodward, were explicitly assumed as a standard by which the book was tested.

(2) With reference to the quotation on page 138 it would appear that since the sentence in question begins "Let us test *the book before us** by this standard, remembering that *its** object is etc.," the significance of the passage is the same whether the words "in this book" be omitted from the quotation or included therein.

(3) The reviewer hopes that the assumption that the book indicates the character of the instruction in secondary technical schools *is* unwarranted, but contends that a form of instruction which for the sake of convenience in testing the accuracy of the pupil's arithmetical work finds "no harm in getting a computed result to thousandths of an inch," when that result is the length of a jack-screw bar, effectually deprives the pupil of any real insight "into the meaning of a certain mathematical formula," and is pernicious formalism in that it does not compel the pupil to consider the significance and reasonableness of the numerical data and results.

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* The italics were not in the review as printed. They seemed unnecessary.

(4) The jack-screw problem is said to be "a problem based on facts." Granted. Also "it was . . . not intended to be a complete and actual problem at all," and it is implied that one would be "lacking in common sense" "to consider such a computed result as finding actual application in the practice of raising buildings." Exactly. That was why it was chosen for criticism. The book aims to show the pupil "what the shop problems are."*

(5) The reviewer is accused of "gross misstatement" of the problem on spiral gearing and malicious intent on his part is implied. The reviewer would point out that of the four results above mentioned, one, the lead of the spiral, is not even mentioned in the solution of the problem as given by the author of the book. The other three quantities may be determined by the formulas: number of teeth $= N = Dp \cos Y$, blank diameter $= D + 2/p$, circular pitch $= \pi/p \cos Y$. Whether any single one or all three of these quantities are to be found from the given data, the determination does not involve operations equivalent to multiplication followed by division by π. This is true also of the determination of the dimensions of the driven gear, and of the numbers of the milling cutters to be used, both of which are given in the book, though not mentioned above. The question at issue is not one of "short cuts" in computation, but of comprehension of the principles and formulas underlying the computation. The reviewer sees no cause to retract any portion of his criticism of the problem.

CHARLES N. HASKINS.

SHORTER NOTICES.

Arithmétique générale. Par EMILE DUMONT. Paris, A. Hermann et Fils, 1911. 8vo. xvii+275 pp. 10 francs.

THIS volume presents a treatment of certain number fields and the ordinary laws of operation within these fields. The book is divided into four parts, of which the first treats of natural numbers, the second of qualified numbers (nombres qualifiés), the third of complex numbers, and the fourth of quaternions and ternions. The treatment is elementary and no attempt is made to extend the subject or to present novel results. The preface, which by the way is one of the most interesting parts of this book, characterizes it quite fully and justly. We quote: "The definitions and the properties of numbers are generally studied by the young student at quite different times and from different classical works. . . . In all this there is no unity of method or point of view. . . . I have therefore attempted to furnish a treatment which should be of use to him who has finished his preliminary studies and who wishes to take a retrospective view of mathematical principles before he launches himself into the vast fields of mathematics and physics."

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* Introduction, p. viii.

Mathematicians are classified as "logicians" and "rationalists." "In analysis the logicians reason on the written characters and in geometry they reason on words." "The rationalists consider mathematics as a preface to physics. They are constantly guided by reason based on experience."

The author follows throughout what he regards as the program of the rationalists. In his hands the fundamental proposition of mathematics (axioms, primitive propositions, or whatever we may call them) are corollaries of physics or of intuitional geometry. Numbers are used to express relations between "magnitudes." These "magnitudes" have properties which we obtain from direct experience with them and these properties we express numerically. The general properties of numbers are determined by the service which they are required to render. The notion of axioms and undefined symbols finds no place in this treatment. There in not an unproved proposition in the book. All the statements which we are wont to regard as assumptions appear under the caption of theorems. The proofs consist in references to remarks on the nature of magnitudes.

From the point of view of the rigorist this treatment is impossible. The reviewer must confess however that there is a certain charm about its naïvété. It is refreshing to see this attempt to set mathematics directly and consciously into life and to validate its elementary propositions directly from a wealth of experience. When the true function of the rigorist is more generally understood we shall perhaps have more patience with this sort of mathematics. With the rigorist the question at issue is frequently not whether the proposition which apparently is being considered is *true* or not, but whether it follows from some other proposition. That is, how much must we say to include by implication a certain other body of propositions.

It seems equally legitimate to inquire what must be the properties of systems of numbers in order that they shall express conveniently and accurately the varied phenomena that daily impinge upon us.

The prospective reader must judge for himself whether this is the sort of book he wants to read. Does he wish to study the modern development of the various algebras as based upon definite assumptions? Then this book is of no use to him. Does he wish to see an attempt to develop these algebras as corollaries of physics? Then it is probably the best book he could find.

N. J. LENNES.

Grundlagen der Geometrie. Von Dr. FRIEDRICH SCHUR. Leipzig, Teubner, 1909. vii + 192 pp.

THE Neuere Geometrie of Pasch marked the first effort to set up a complete set of fundamental statements for geometry —if we except the manifoldness development of Riemann. Following the appearance of that book there came a remarkable development of the subject that has thrown great light upon the logic of geometry. Schur wishes the present book to be considered as, in a sense, a revision of Pasch. Although the author expresses very strongly his indebtedness to Pasch, in only the most general way can the book be said to be such a revision.

In the general trend of his development the author follows Peano, in that congruence is obtained from motion or from projective geometry rather than directly from postulates, as is done by Hilbert, for example.

The general problem is formulated as follows: "To set up a simple and complete system of intuitive facts or axioms, entirely independent of one another, from which geometry can be derived by purely logical processes. To deserve the name geometry, axioms must be employed which express the results of the simplest and most elementary consideration of geometric figures, from which they are obtained by abstraction." This of course bars out the idea of space as a number manifold.

Although in a general way the effort is made to have the axioms independent, there would seem to be no effort to have each axiom contain but a single statement. In this respect the book lacks the elegance of logical form of the work of Veblen.

In § 1, the "projective postulates" are developed. The point and the segment (Strecke) are employed as undefined symbols, and the postulates concerning them in this chapter are sufficient to develop the ordinary connective properties of space and the order relations of points, and to characterize the space as being of exactly three dimensions. These axioms are equivalent to the first two groups of axioms of Hilbert. The line, plane, and space are formally defined in terms of point and segment, and the separation theorems are proved.

In § 2, after proving Desargues's theorem concerning perspective 3-edges in a bundle of lines in 3-space, it is shown that a line and a plane not containing it, or two lines in the same plane, determine a set of lines having the properties of a bundle. If the existence of the point of intersection is not certain, the bundle thus obtained is said to determine an ideal point. Similarly ideal lines are defined by means of non-intersecting planes. Desargues's theorem is then proved in the plane, and the complete duality of the system of actual and ideal points is shown. That the Desargues theorem can not be proved in the plane alone is shown by means of the non-desarguesian geometry of Moulton. Finally an ideal plane is defined as the locus of lines which join an ideal point to all points of an ideal line not containing it, and the ideal connection properties of 3-space are then complete. Since the only assumption concerning the intersection of lines in a plane is that regarding the points of a triangle, i. e., a restricted domain, the development can serve as a basis for the euclidean and non-euclidean geometries which differ regarding intersection properties of two lines in the unlimited plane. The central collineation and collinear reflections are then defined, and used for the definition of harmonic points.

In § 3, postulates are introduced to make precise the concept of motion, and on this are based definitions of absolute pole and polar, perpendicular lines and planes, folding, and rotation, and the proof of Pascal's theorem for the degenerate conic. Finally congruence of segments is defined as follows: Two segments are called congruent, if one can be transformed into the other by a motion.

In § 4, the fundamental theorem of projective geometry is proved and the algebra of segments and resulting analytic geometry are developed. The fundamental theorem is proved by making use of the postulates of motion, a procedure that appears less natural from the projective point of view than it would be to choose as a postulate either this theorem itself, or the theorem of Pascal.

The projective development is then applied for the derivation of the fundamental metric forms of non-euclidean geometry. The equations of lines are obtained, and the trigonometric functions sine and cosine are defined projectively. The theory of triangles then follows readily. The characteristic constant k occurring in these formulas depends on the character of the absolute involution obtained on each line. Congruence of triangles is developed, and construction of the triangle from given parts. The proof of the independence of the parallel axiom is made by exhibiting in the classical way the analytic number spaces for which each hypothesis concerning parallels is valid. The relation of the parallel axiom to the angle-sum of a triangle and to motion is also discussed. In § 7, plane geometry is developed without the use of 3-space, following the methods of Hjelmslev. In § 8, the role of the Archimedean axiom is treated. The proof that while the Pascal theorem does not follow from the projective postulates alone, it does follow from these with the addition of the Archimedean postulate, is especially interesting.

As a whole the book is a valuable addition to the literature of the subject. More references to original sources would add much to its value. It is not free from typographical errors, but those noted are fairly evident from the context.

F. W. OWENS.

Précis de Mécanique rationnelle. Introduction à l'Étude de la Physique et de la Mécanique appliquée. Par P. APPELL et S. DAUTHEVILLE. Paris, Gauthier-Villars, 1910. 8vo. 729 pp. 25 francs.

To condense an enormous Traité in three volumes and over 1,800 pages into a *précis*—a very significant term—of one volume and about 700 pages, demands a treatment which is more than a mere deletion of certain parts. The whole must to some extent be recast. Everyone has seen abridged treatises which were worthless and by the side of the original quite obscure. In the present case however the task has been successfully done, and we have in this short course an admirable treatise on mechanics which covers a very wide range in a thorough manner. As the title indicates, the subject considered is rational mechanics, yet it is brought so close to applied mechanics that the passage from one to the other is very easy. Although the standpoint of the text is modern, yet it is above all a book for the undergraduate student.

The object of mechanics is stated to be the solution of two problems:

To find the motion of a system of bodies under the action of given forces;

To find the forces that will produce a given state of motion. By force, it is carefully explained, is meant a purely fictitious cause, not a real physical cause. It is merely an abbreviation for the product of acceleration by mass. Mass is defined to be the arbitrary number that may be assigned to a material point, such that the inverse ratio of two such arbitrary numbers is equal to the ratio of the accelerations they produce in each other. Thus all problems are reduced to questions of acceleration, that is, to questions of functions of four variables x, y, z, t and the derivatives of the first three as to the fourth. In the preface of the second edition of the third volume of the Traité, attention is called to a note at the end of the volume on "la théorie de l'action euclidienne." The object of this note is to reduce the whole theory of mechanics to the consideration of certain invariants in euclidean displacements. The note is not repeated in the Précis, but the spirit of it is in evidence throughout. The development of the course, however, does not demand on the part of the student a familiarity with group theory nor even great knowledge of analysis. Some acquaintance with differential equations is necessary. Intuitive methods are used to some extent and particular problems are developed in detail usually, in order to make the use of general theorems quite clear. The ripe experience of the authors shows itself continually, so that one feels that he has here a finished product of many years preparation. We will undertake to indicate some of the main features in some detail.

There are three parts: Preliminary Notions, Statics, and Dynamics. In the first we find chapters on vectors, kinematics, and the principles of mechanics; in the second, on equilibrium of points, of solids, of deformable systems; in the third, on dynamics of a point, moments of inertia, the seven universal equations of motion, motion of a solid, friction, impact, virtual work, d'Alembert's principle and Lagrange's equations, attraction and potential, hydrostatics, and hydrodynamics. A selection of examination problems closes the book. The omissions are in general of the more elaborate problems or the more difficult parts of the theory.

Although, as seems to be the more common custom in France, the notation of vectors is not used in this course, nevertheless the first chapter is devoted to a sketch of the different kinds of vectors and their more important theorems. Three classes are noted: the free vectors, the sliding (glissant) vectors, and the bound vectors. The terminology of these is freely used throughout the course. The first class include those which combine according to the parallelogram law, such as translations, velocities, accelerations, axes of couples, and like quantities. The second class comprise such quantities as forces acting on a rigid body, linear moments, moments of momentum, screws, etc. The third class are the localized vectors, such as a velocity field, acceleration field, electromagnetic field, vortex field. The distinction between axial and polar vectors is also explained. Two invariants are noted: $X^2 + Y^2 + Z^2$ and $LX + MY + NZ$. The first is the square of the length of the resultant, the second represents the sum of the volumes of all the tetrahedra made by taking the sliding vectors in pairs as the opposite edges of tetrahedra.

The chapter on kinematics has been doubled in length over that in the Traité. This is partly from the examples introduced, but it is also due to the very much more detailed treatment of the definitions and the theorems. There is a distinct gain for the elementary student. The development produces the usual expressions for the velocity and the acceleration in various coordinates, discusses the motion of a solid, and the problems of relative motion. If one were to offer a suggestion on this part of the book, it would be that a brief consideration of relativity in general might not be amiss.

The chapter on the principles of mechanics lays down the following as the foundation of what follows:

1. Inertia: every material point which is supposed to be isolated has no acceleration.

2. Two material points moving with any velocities produce in each other oppositely directed accelerations along the line

joining the two points. (The action may be due to electricity, gravity, magnetism, etc.)

3. The ratio of the numerical values of these accelerations is constant, that is, the ratio is independent of the physical conditions of the points, or of their motion. These ratios are inversely as the ratios of the masses, one mass being purely arbitrary. Masses are analogous to chemical equivalents.

4. The accelerations of a point due to several points compound according to the parallelogram law.

One might suggest that here a brief note or paragraph could have been given on the definitions of mass which depend on the velocity, and the place of the "new mechanics" with regard to the "Newtonian mechanics." Such an explanation would orient the student as to some of the more recent work in mechanics.

The notion of force function and field of force are introduced in connection with the idea of work. The principal fields are mentioned: uniform field, gravity field, electric field, magnetic field.

The general conditions of equilibrium are stated in the succinct form: That any system be in equilibrium it is necessary and sufficient that the forces form a system of sliding vectors equivalent to zero. In the discussion of the equilibrium of a solid body we miss the single condition given in the Traité: for the equilibrium of a solid body the necessary and sufficient condition is that the sum of the moments of all the forces with respect to each of the edges of any tetrahedron vanish for each edge. The simplicity of this vector statement can scarcely be improved. It is also very general, as the tetrahedron may be chosen to suit the investigation. The discussion of the subject of the chapter is more detailed than in the Traité, with an increase of clearness and simplicity. The consideration of virtual work has been transferred to a later chapter.

Chapter VII, opening the third part, is devoted to the dynamics of a point. These topics are discussed: general theorems, straight motion, curvilinear motion of a heavy point in a vacuum and in a resisting medium, electrified particle in a field, motion of a point on a surface, equilibrium and relative motion. The text follows the Traité, the more special or complicated cases being omitted. The illustrative examples are carefully worked out and discussed.

The chapters on Lagrange's equations, d'Alembert's principle, and canonical equation have been placed farther on, after the treatment of virtual work.

Chapter IX considers the dynamics of systems, general theorems, and the seven universal equations of motion. These theorems include the conservation of the motion of the center of gravity of the system, of the sum of the moments of momentum, and of vis viva. These lead to seven general equations of motion, three for the center of gravity, three for the moments of momentum, one for the kinetic energy. The internal forces do not enter the first six, but are present in the last. It is shown that these equations remain unchanged if the coordinates are referred to axes moving parallel to themselves with a uniform motion, or to axes moving parallel to themselves and always having the center of gravity as origin. It is pointed out that the motion in any case is reducible to the motion of the center of gravity plus that of the parts of the system with reference to the center of gravity. Thus the question of relative and absolute motion enters again. The meaning of energy and the law of conservation of energy close the chapter.

Chapter X takes up the motion of a solid body. The discussion is divided into the study of motion about an axis, motion parallel to a plane, screw motion, motion about a fixed point, motion about a point in a field which is directed to the point, motion about a fixed point in a gravity field, and motion of a free solid. The particular problems worked out are the composite pendulum, a homogeneous straight wire passing through a fixed point and sliding in a plane, a heavy plate sliding on a line in a plane. The Traité considers several other problems, some of which might have come in here: the Atwood machine, wheel rolling down an incline, double cone, the hoop, the top.

Friction is considered in the following chapter. It is practically defined as the effect of a force tangential to the surface, and two couples, one whose axis is the normal, and one whose axis is tangent. There are thus friction of sliding, of spinning, and of rolling. The spinning friction is omitted from the discussion.

Chapter XIII is devoted to developing the principle of virtual work which gives rise to what is here rather technically called analytic mechanics. From this principle are deduced the general conditions of equilibrium, and its importance is pointed out for that treatment of dynamics which reduces the problems to problems in statics. The next chapter takes up the principle of d'Alembert, which is made the sole basis of the development. The principles of Gauss and of Hamilton and the principle of least action, are given in the Traité but omitted here. Lagrange's equations, the distinction between holonome and non-holonome systems, small movements about a position of equilibrium, and the canonical equations are discussed. These are all illustrated with examples worked out in full and with practical directions as to the use of the theorems. It is also suggested that many of the problems could as well have been solved by methods previously given. The next chapter completes the treatment of an earlier one on impact. A method analogous to the principle of d'Alembert is used.

In Chapter XVI we find Green's theorem, stated in the usual way and also in the vector form: the integral of the flux of the field through a closed surface is equal to the integral of the divergence of the field over the points enclosed by the surface. This theorem is then applied to the theory of attraction and potential.

The last chapters are concerned with the equilibrium of fluids, and with hydrodynamics. Both Lagrange's and Euler's methods are developed, and there is a brief treatment of vortices.

Taken as a whole, it is difficult to see how the text could be better. The book has a finished character, and is so practical as a text that one lays it down well pleased with its form and contents.

JAMES BYRNIE SHAW.

Systèmes cinématiques. Par L. CRELIER. Paris, Gauthier-Villars, 1911, 99 pp.

THIS monograph is one of the recent numbers of the well-known French collection Scientia. The author confines himself to the solution of a number of problems in generating plane algebraic curves by link motions, so that the general title of kinematic systems for the ground covered, even when limited to the plane, seems not entirely justified.

We should expect an introductory chapter on the general principles of link motions and the transformations realized by them. The theorems of Kempe and Koenigs concerning the realization of all algebraic curves and surfaces by linkages should at least be mentioned. A treatment of general propositions like these would be of far greater importance than the solution of any number of particular problems without correlation.

Among the list of references we miss the names of Burmester, Somoff, and in view of the subject matter treated, De Ross, who in the *Revue Universelle des Mines* published a series of articles on linkages, their different forms and uses. A translation of these articles appeared 1879 in Van Nostrand's Science Series. As in Crelier's kinematic systems, the tracing of curves is the main object of De Ross's investigations. While De Ross uses Peaucellier's inversor and its modifications as a generator, Crelier does not mention it at all.

The first three chapters deal with the curves which are produced by points and lines in connection with the motion of a right angle subject to certain conditions. In the next chapter conchoidal circular systems and curves are studied. The displacement of the extremities of a straight line of constant length along two rectangular lines, and on a straight line and a circle are the subjects of the concluding two chapters.

A number of new curves are obtained from the various motions. The treatment is throughout clear and simple and does not require more than the elements of plane analytic geometry and calculus. In some parts the typography is poor and a number of figures lack precision and neatness of execution.

ARNOLD EMCH.

NOTES.

AT the meeting of the London mathematical society held on January 11 the following papers were read: By W. H. Young, "Successions of integrals and Fourier series"; by G. H. Hardy and J. E. Littlewood, "A new condition for the truth of the converse of Abel's theorem"; by A. Cunningham, "On Mersenne's numbers."

AT the meeting of the Edinburgh mathematical society on February 8 the following papers were read: By Professor Goldziher, "On graphical integration"; by Dr. Muirhead, "A mechanism for solving equations of the nth degree"; by G. Philip, "The geometry of the general pedal curve"; by W. Gentle, "An extension of the remainder theorem."

AMONG the prizes to be awarded by the Paris academy of sciences in 1913, the following are for achievements in pure and applied mathematics: The Francoeur prize (1000 fr.) for discoveries or works useful to the progress of pure or applied mathematics; the Bordin prize (3000 fr.) for improving in some important point the arithmetic theory of non-quadratic forms; the Poncelet prize (2000 fr.) for work in applied mathematics; the Lalande prize (540 fr.) in theoretical astronomy; the Pontécoulant prize (700 fr.) in theoretical astronomy; the Saintour prize (3000 fr.) for the most meritorious memoir in mathematics; the Petit d'Ormay prize (10000 fr.) in pure and applied mathematics; the Pierson-Perrin prize (5000 fr.) in mechanics and physics; the Wilde prizes (one of 4000 fr., one of 2000 fr.) for the most important contributions to science made during the two years immediately preceding the time the awards are made. The competition for all these prizes is unrestricted.

The academy has awarded the Saintour prize (3000 fr.) for 1911 to Professor J. DRACH, of the University of Toulouse, for his memoir on "Groups of rationality of differential equations."

THE international commission on the teaching of mathematics announces the following programme for its next session, in connection with the fifth congress of mathematicians at Cambridge: At the general session the president, Professor F. KLEIN, will give a summary of the results already reached, and formally request that the commission be continued until the next congress in 1916. The commission will then hold three sessions in affiliation with section IV of the congress. At the first the reports of the chairmen of the national subcommissions will be received; at the second, the report of committee A of the Milan meeting will be discussed, and at the third, the report of committee B. (See BULLETIN, volume 18, page 36.) Thus far, forty-nine monographs have been published by the various sub-committees, and a number of others will appear during the next few weeks.

THE report of the national committee of fifteen on a geometry syllabus, which has been under consideration for nearly three years, has finally been published in a pamphlet of 70 pages and is ready for distribution to teachers of geometry, and all others interested. This report was prepared under the joint auspices of the American Federation of Teachers of the Mathematical and Natural Sciences and the National Education Association. It includes a historical introduction and sections on axioms and definitions, on exercises and problems, and the syllabus itself including both plane and solid geometry. It is the hope of the committee that this report may be of great service to all teachers of geometry, and to this end that it may have a wide distribution among all interested. Copies may be secured gratis upon application to the Commissioner of Education, Department of the Interior, Washington, D. C.

THE firm of Eimer and Amend, 205–211 Third Avenue, New York City, have just issued an Illustrierter Spezialkatalog mathematischer Modelle und Apparate für den Unterricht in der Planimetrie, Stereometrie, Trigonometrie und verwandten Fächern, entworfen von G. Koepp und anderen bewährten Fachmännern. The catalogue contains besides an index, 128 pages of descriptions and prices of several hundred models, largely for instruction in the secondary schools, and 324 illustrations.

MR. W. M. COATES, fellow of Queen's College, Cambridge, died at Cambridge January 16, at the age of 54 years. Mr. Coates had the reputation of being the most successful mathematical coach since the time of the late Dr. E. J. Routh, under whom he won the third wranglership in 1883.

RECENT catalogues of new and second-hand books:—G. J. Göschen'sche Verlagshandlung, Marienstrasse 18, Leipzig, publications of this firm in mathematics and allied sciences, 1907–1911.—A. Hermann et Fils, 6 rue de la Sorbonne, Paris, 300 titles in mathematics and exact sciences.—E. Beyer, Schottengasse 7, Vienna, catalogue 61, 1296 titles in mathematics, astronomy and meteorology.—G. Fock, Schlossgasse 7–9, Leipzig, catalogue 391, 3500 titles in mathematics and physics, also catalogue 407, 4000 titles in mathematics, physics and astronomy.—J. Schweitzer, Lenbachplatz 1, Munich, catalogue 54, 2300 titles in insurance.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

d'Arcais (F.). Analisi infinitesimale. 3a edizione. Volume 1. Fascicolo 1. Padova, 1911. 8vo. 448 pp. L. 12.00

Bachmann (P.). See Klein (F.).

Baum (K. T.). Der Kreis und seine Quadrate, die arithmetische Wage, $\pi = 3.125$ oder $3\frac{1}{8}$ und die Quadratur des Kreises. Saarbrücken, 1911. 8vo. M. 1.20

Böhm (K.). Axiome der Arithmetik. Heidelberg, 1911. 8vo. 11 pp.

Brendel (M.). See Klein (F.).

Bücher, Neue, über Naturwissenschaften und Mathematik. (Die Neuigkeiten des deutschen Buchhandels, nach Wissenschaften geordnet.) Mitgeteilt Herbst 1911. Leipzig, Hinrichs, 1911. 8vo. 68 pp. M. 0.30

Cain (W.). A brief course in the calculus. 3rd edition, revised. New York, Van Nostrand. 8vo. 11+281 pp. $1.75

Coffin (J. G.). Vector analysis. An introduction to vector-methods and their various applications to physics and mathematics. 2nd edition. New York, Wiley, 1911. 12mo. 22+262 pp. Cloth. $2.50

Cohen (A.). An introduction to the Lie theory of one parameter groups. New York, Heath, 1911. 12mo. 247 pp. $2.00

Dini (U.). Sugli sviluppi in serie per la rappresentazione analitica delle funzioni di una variabile reale date arbitrariamente in un certo intervallo. Pisa, 1911. 8vo. 480 pp. L. 12.00

Dixon (A. C.). On the series of Sturm and Liouville, as derived from a pair of fundamental integral equations instead of a differential equation. London, Dulau, 1912. 4to. 22 pp. 1s.

Egoroff (C.). Le théorème de Fermat. St. Petersburg, 1911. 8vo. 18 pp. M. 3.00

Ehrenberg (H.). Kants mathematische Grundsätze der reinen Naturwissenschaften. Heidelberg, 1910. 8vo. 60 pp.

Fagnano (G. C.). Opere Matematiche. Milano, 1911. 8vo. L. 45.00

Gambier (G.). Le mathematicien François Viète. Généalogie de sa famille. La Rochelle, 1911. 8vo. 31 pp.

Haeussler (J. W.). Allgemeiner Beweis des Fermatschen Satzes. Berlin, Krayn, 1912. 8vo. 20 pp. M. 0.80

Heidelberg (P.). Allgemeiner Beweis des Fermatschen Satzes. 3te Auflage. Coblenz, Erben, 1911. 8vo. 3 pp. M. 0.60

Heffter (L.). Über eine vierdimensionale Welt. Freiburg, 1911. 8vo. M. 0.90

Holst (E.). See Matematikken.

Karnash (D.). Beweis für den Fermatschen Satz. Berlin, Mayer und Muller, 1911. 8vo. 17 pp. M. 1.20

Katalog mathematischer Modelle für den höheren mathematischen Unterricht. Leipzig, Schilling, 1911. 8vo. 14+172 pp. M. 1.20

Kelvin (Lord). Mathematical and physical papers. Vol. 6. London, Cambridge University Press, 1912. 8vo. 386 pp. 10s.

Kiepert (L.). Grundriss der Differential- und Integral-Rechnung. 2 Teile. 1: Differential-Rechnung. Hannover, Helwing, 1912. 8vo. 20+863 pp. M. 13.50

——. Grundriss der Differential- und Integral-Rechnung. 2 Teile. 2: Integral-Rechnung. Hannover, Helwing, 1912. 8vo. 23+738 pp. M. 13.50

Klein (F.) und Brendel (M.). Materialen für eine wissenschaftliche Biographie von Gauss. Heft 1: Über Gauss' zahlentheoretische Arbeiten (von P. Bachman). Leipzig, Teubner, 1911. 8vo. 3+54 pp. M. 2.00

Koch (W.). Beiträge zur affinen Geometrie der Flächen zweiten Grades. Kiel, 1910. 8vo. 54 pp.

Krause (M.). Theorie der elliptischen Funktionen. Leipzig, 1912. 8vo. 6+186 pp. M. 3.60

Levy (P.). Sur les équations intégro-différentielles définissant des fonctions de lignes. (Thèse.) Paris, Gauthier-Villars, 1911. 4to. 121 pp.

Lietzmann (W.). Der pythagoreische Lehrsatz, mit einem Ausblick auf das Fermatsche Problem. Leipzig, Teubner, 1911. 72 pp. M. 0.80

Löffler (E.). Die Ziffern und Ziffernsysteme der Kulturvolker in alter und neuer Zeit. Leipzig, Teubner, 1911. 84 pp. M. 0.80

Massenet (G.). Eléments de calcul infinitésimal. Paris, Challamel, 1912. 8vo. 8+292 pp.

Masurkewitz (J.). Über die Grenzen der Lösbarkeit der Gleichung $x^\lambda + y^\lambda = z^\lambda$ in ganzen Zahlen. (Grosser Fermatscher Satz.) Berlin, 1911. 8vo. 4 pp. M. 0.50

Matematikken og Naturvidenskapen ved det K. Fredriks Universitet, 1811–1911. Sårtryk av Festskrift. Matematikken. av E. Holst. Kristiania, 1911. 4to. 151 pp. M. 6.50

Milne (W. P.). Projective geometry. New York, Macmillan, 1911. 12mo. 15+148 pp. $0.75

Niewenglowski (B.). Cours de géométrie analytique. Tome 2: Construction des courbes planes. Compléments relatifs aux coniques. Courbes définies par les équations différentielles. 2e édition. Paris, Gauthier-Villars, 1912. 8vo. 4+328 pp. Fr. 9.00

Paoletti (G.). Applicazioni della teoria di Lie sui gruppi continui di trasformazioni alla risoluzione delle equazioni differenziali. Rocca S. Casciano, 1911. 8vo. 64 pp. L. 2.25

Puzyna (J.). Über Systeme von Kurven mit der Gruppe pseudolinearer Substitutionen. (Polish.) Krakau, 1911. 8vo. 46 pp.

Schubert (H.). Niedere Analysis. (Sammlung Schubert. Band 45.) 2ter Teil: Funktionen, Reihen, Gleichungen. 2te durchgesehene Auflage. Leipzig, Göschen, 1911. 8vo. 5+215 pp. Cloth. M. 3.80

Sterzinger (O.). Zur Logik und Naturphilosophie der Wahrscheinlichkeitslehre. Leipzig, 1911. 8vo. M. 4.50

STOFFAES (E.). Cours de mathématiques supérieures à l'usage des candidats à la licence ès sciences physiques. Premier tome: Compléments d'algèbre élémentaire. Dérivées. Equations. Géométrie analytique. Différentielles et intégrales. Paris, Gauthier-Villars, 1911. 8vo. 6+398 pp. Fr. 10.00

SUINI (A.). Delle definizioni di retta e di piano quale vere basi della geometria: studio di filosofia matematica, a complemento dell'altro contenuto nell'opuscolo dal titolo: La confutazione della geometria non euclidea e la teoria naturale delle parallele. Piacenza, Porta, 1911. 8vo. 18 pp. L. 1.00

TURRIÈRE (E.). Sur les congruences de normales qui appartiennent à un complexe donné. (Thèse.) Toulouse, Privat, 1911. 4to. 88 pp.

WIELEITNER (H.). Geschichte der Mathematik. (Sammlung Schubert. Band 63.) 2ter Teil: Von Cartesius bis zur Wende des 18ten Jahrhunderts. 1te Hälfte: Arithmetik, Algebra, Analysis. Bearbeitet unter Benutzung des Nachlasses von A. v. Braunmuhl. Leipzig, Göschen, 1911. 8vo. 8+251 pp. Cloth. M. 6.50

II. ELEMENTARY MATHEMATICS.

ADLER (A.) und MÜLLER (E.). Berichte über den mathematischen Unterricht in Österreich. Band 9. 1ter Teil: Der Unterricht in der darstellenden Geometrie an den Realschulen und Realgymnasien (von A. Adler). 2ter Teil: Der Unterricht in der darstellenden Geometrie an den technischen Hochschulen (von E. Müller). Wien, Hölder, 1911. 8vo. 124 pp. M. 2.40

BOUVART (C.) et RATINET (A.). Règles et formules usuelles servant de supplément aux tables de logarithmes, contenant: 1, une liste des nombres premiers et des nombres n'ayant pas de diviseur moindre que 13; 2, les racines carrées des nombres de 1.0 à 9.9 et de 10 à 100; 3, les longeurs des arcs et des lignes trigonométriques dans les deux systèmes de division; 4, les règles et les formules usuelles des cours de mathématiques et de physique; 5, des tableaux relatifs aux poids atomiques, aux densités et aux températures de fusion et d'ébullition. 3e édition, revue. Paris, Hachette, 1912. 8vo. 47 pp. Fr. 0.60

BRANDENBERGER (K.). Der mathematische Unterricht an den schweizerischen Gymnasien und Realschulen. Der mathematische Unterricht in der Schweiz. Band 4. Basel, Georg, 1911. 8vo. 167 pp. M. 3.50

CHINI (M.). Lezioni di algebra elementare. Seconda edizione del I volume delle Lezioni di algebra, ad uso dei licei. Livorno, Giusti, 1912. 8vo. 8+171 pp. L. 1.80

DIETSCH. See SIEVERT.

FERVAL (H.). Eléments de trigonométrie, rédigés conformément aux programmes de l'enseignement secondaire et de l'enseignement primaire supérieur. 2e édition, revue et corrigée. Paris, Hachette, 1911. 16mo. 664 pp. Fr. 5.00

F. G. M. Exercices d'algèbre. 7e édition. Paris, Gigord, 1912. 8vo. 24+955 pp.

FRINK (F. G.). See HALL (A. G.).

Gajdeczka (J.). Lehrbuch der Geometrie für die oberen Klassen der Mittelschulen. 2te Auflage. Nach den Lehrplänen von 1909 umgearbeitet von E. Kaller. Wien, Deuticke, 1911. 8vo. 5+267 pp. Cloth. M. 3.60

——. Übungsbuch zur Geometrie für die oberen Klassen der Mittelschulen. 4te Auflage. Nach den Lehrplänen von 1909 umgearbeitet von E. Kaller. Wien, Deuticke, 1911. 8vo. 5+225 pp. Cloth. M. 2.80

Garneri (A.). Corso elementare di disegno geometrico. Parte I: Problemi grafici geometrici e ornamentazione geometrica. 24a edizione. Torino, Paravia, 1911. 16mo. 6+134 pp. L. 2.00

——. Corso elementare di disegno geometrico. Parte II: Proiezioni ortogonali e applicazione alla meccanica, teoria delle ombre, prospettiva parallela. Settima edizione, completamente rifatta. Torino. Paravia, 1911. 16mo. 159 pp. L. 2.00

Gauss (F. G.). Fünfstellige vollständige logarithmische und trigonometrische Tafeln. Zum Gebrauche fur Schule und Praxis bearbeitet. Stuttgart, Wittwer, 1912. 8vo. 31+177 pp. Cloth. M. 2.50

Graf (J. H.). See Lacombe (M.).

Grossmann (M.). Der mathematische Unterricht an der eidgenössischen technischen Hochschule. (Der mathematische Unterricht in der Schweiz. Band 7.) Basel, Georg, 1911. 8vo. 52 pp. M. 2.00

Hall (A. G.) and Frink (F. G.). Plane and spherical trigonometry. New York, Holt, 1911. 8vo. 10+176 pp. $1.00

Hall (W.). Tables and constants to four places for use in technical and nautical computation. New York, Putnam, 1911. 8vo. 9+60 pp. $0.90

Hartl (H.). Übungsbuch für den Unterricht in der allgemeinen Arithmetik und Algebra an Werkmeisterschulen, Baugewerkschulen und verwandten Lehranstalten. Wien, Deuticke, 1911. 8vo. 6+163 pp. Cloth. M. 2.40

Hobson (E. W.) A treatise on plane trigonometry. 3rd edition. London, Cambridge University Press, 1912. 8vo. 400 pp. 12s.

Huntington (E. V.). Four place tables of logarithms and trigonometric functions. London, Spon, 1912. 8vo. 3s.

Jacob (J.) und Schiffner (F.). Arithmetik und Geometrie für Gymnasien und Realschulen. Geometrie des Raumes. 2ter Teil. Wien, Deuticke, 1911. 8vo. 3+114 pp. M. 2.00

——. Planimetrie und Stereometrie. Für die 4te und 5te Klasse. Wien, Deuticke, 1911. 8vo. 198 pp. Cloth. M. 2.80

Lacombe (M.) und Graf (J. H.). Der mathematische Unterricht in der Schweiz. Band 8. 1ter Teil: L'enseignement mathématique à l'école d'ingénieurs de Lausanne (von M. Lacombe). 2ter Teil: Der mathematische Unterricht an den schweizerischen Universitäten (von J. H. Graf). Basel, Georg, 1911. 8vo. 72 pp. M. 2.25

Macé de Lepinay (A.). See Vacquant (C.).

Müller (E.). See Adler (A.).

Nicholson (J. W.). Key to school algebra. New York, American Book Company, 1911. 77 pp. $0.50

NOODT (G.). Mathematische Experimentiermappe für den geometrischen Anfangsunterricht. Leipzig, Teubner, 1912. 4+44 pp. M. 4.00

QUESTIONS and solutions, mathematical, from the Educational Times. Edited by C. J. Marks. London, 1911. 8vo. Cloth. 7s.

RATINET (A.). See BOUVART (C.).

RIBONI (G.). Elementi di geometria, ad uso delle scuole secondarie superiori. Sesta edizione, con aggiunte e correzioni. Bologna, Zanichelli, 1911. 16mo. 523 pp. L. 3.50

SCHIFFNER (F.). See JACOB (J.).

SIEVERT und DIETSCH. Lehrbuch der Geometrie. 1: Planimetrie. 5te Auflage. Lösungen der Aufgaben. Leipzig, Deichert, 1912. 8vo. 42 pp. M. 1.00

SOCCI (A.) e TOLOMEI (G.). Elementi di geometria secondo il metodo di Euclide: libro di testo per i ginnasi ed i licei. Volume II, per la 1a classe liceale. 8a edizione. Firenze, Niccolai, 1911. 8vo. 200 pp. L. 2.00

SPEROTTI (E.). I logaritmi per i ragionieri: tavole dei logaritmi volgari a otto decimali dei numeri interi da 1 a 100,000. Rocca S. Casciano, Capelli, 1911. 16mo. 154 pp. L. 2.50

SPIELMANN (J.). Lösungen der Aufgaben von Močniks Lehrbuch der Geometrie fur die 4-8 Klasse der Gymnasien und Realgymnasien und für die 4-7 Klasse der Realschulen. Wien, Tempsky, 1912. 8vo. 74 pp. M. 1.60

TOLOMEI (G.). See SOCCI (R.).

VACQUANT (C.) et MACÉ DE LEPINAY (A.). Eléments de géométrie à l'usage des classes de sciences (second cycle). Paris, Masson, 1912. 16mo. 672 pp. Fr. 5.25

——. Premières notions de géométrie élémentaire à l'usage des classes de lettres. 2e partie: Géométrie dans l'espace. Paris, Masson, 1912. 16mo. 6+176 pp. Fr. 1.50

III. APPLIED MATHEMATICS.

BAKKER (G.). Théorie de la couche capillaire des corps purs. Volume 1: Théorie de la couche capillaire plane des corps purs. Paris, Gauthier-Villars, 1911. 8vo. 96 pp. Fr. 2.00

BÉCOURT (L.). Dessin technique. Cours professionnel de dessin géométrique. 7e édition. Paris, Hatchette, 1911. 16mo. 16 pp. Fr. 1.00

BEHRENS (W.). Ein der Theorie der Laval-Turbine entnommenes mechanisches Problem, behandelt mit Methoden der Himmelsmechanik. (Diss.) Göttingen, 1911. 8vo. 58 pp.

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CARPENTER (R. C.) and DIEDERICHS (H.). Experimental Engineering. New York, Wiley, 1911. 8vo. 19+1132 pp. $6.00

CASTELLANO (F.). Lezioni di meccanica razionale. Torino, 1911. 8vo. 470 pp. L. 10.60

Catalogue, international, of scientific literature. Mechanics. London, Royal Society, 1911. 8vo. 156 pp. 10s. 6d.

Diederichs (H.). See Carpenter (R. C.).

F. G. M. Cours de mécanique, suivi d'un recueil de problèmes pour la classe de mathématiques. Paris, Poussielgue, 1912. 8vo. 6+436 pp.

Föppl (A.). Graphische Statik. (2ter Band, Vorlesungen über technische Mechanik.) 3te Auflage. Leipzig, Teubner, 1912. 12+419 pp. M. 8.00

Fourquet (J.). Eléments de géométrie descriptive appliquée au dessin et à la construction. 1er tome: Modelerie, forge, ajustage. 2e tome: Coupe des pierres, charpente et menuiserie. Lille, Janny, 1912. 8vo. Tome 1, 148 pp. Tome 2, 159 pp.

Gordon (J.). See Gray (A.).

Gray (A.) and Gordon (J.). A treatise on dynamics. New York, Macmillan, 1911. 12mo. 16+626 pp. $3.25

Groos. Anwendung der Wahrscheinlichkeitstheorie auf dem Gebiet der Schiesslehre. Berlin, 1911. 8vo. M. 8.00

Hager (K.). Berechnung ebener, rechteckiger Platten mittels trigonometrischer Reihen. München, Oldembourg, 1911. 8vo. 7+94 pp. M. 7.20

Hartley (D. H.). Analysis of the law of insurance. London, 1911. 8vo. Cloth. 3s.

Helmholtz (H. von). Vorlesungen über theoretische Physik. Herausgegeben von A. König, O. Krigar-Menzel, F. Richarz, C. Runge. 2te Abteilung: Vorlesungen über die Dynamik discreter Massenpunkte. 2te Auflage. Leipzig, Barth, 1911. 8vo. 11+380 pp. M. 15.00

Hopf (L.). Hydrodynamische Untersuchungen: Turbulenz bei einem Flusse. Über Schiffswellen. (Diss.) München, 1910. 8vo. 91 pp.

Izart (J.). Mécanique. À l'usage des ingénieurs, constructeurs-mécaniciens, industriels, chefs d'ateliers, et contremaîtres. 34e édition. Paris, Dunod et Pinat, 1912. 16mo. 7+340 pp. Fr. 3.00

Karapetoff (V.). Engineering applications of higher mathematics. New York, Wiley, 1912. 12mo. 15+69 pp. $0.75

Liepe (S.). Die Verwendung der Brinellischen Kugeldruckprobe zu Kraft- und Schlagarbeitsmessungen. (Diss.) Aachen, 1911. 4to. 52 pp.

Loppé (F.). Essais industriels des machines électriques et des groupes électrogènes. 2e édition. Paris, Gauthier-Villars, 1911. 8vo. 317 pp. Fr. 10.00

Noeggerath (J. E.). Über die Stromabnahme mit besonderer Berücksichtigung hoher Geschwindigkeiten. (Diss.) Hannover, 1911. 4to. 17 pp.

Orlich (E.). Die Theorie der Wechselströme. (Mathematisch-physikalische Schriften für Ingenieure und Studierende. Band 12.) Leipzig, Teubner, 1912. 8vo. 4+94 pp. Cloth. M. 2.80

Randall (J. A.). Heat; a manual for technical and industrial students. New York, Wiley, 1911. 12mo. 127 pp. $0.50

ROUMAJOU (J.) et SILVESTRE (E.). Mécanique, d'après les programmes officiels du 27 juillet 1909. Paris, Delagrave, 1911. 18mo. 7+411 pp. Fr. 3.75

SCHIFFNER (F.). Leitfaden für den Unterricht in der darstellenden Geometrie an österreichischen Oberrealschulen und verwandten Lehranstalten. Wien, Deuticke, 1911. 8vo. 6+246 pp. Cloth. M. 3.60

SCHULZE (P.). Allgemeine Theorie unsymmetrischer Schwingungen und Ablenkungen bei Systemen mit einem Freiheitsgrad und Anwendung derselben auf das zu erdmagnetischen Messungen dienende Unifilar-Magnetometer. (Progr.) Gardelegen, 1911. 8vo. 29 pp.

SILVESTRE (E.). See ROUMAJOU (J.).

SPARRE (LE COMPTE DE). Sur le mouvement des projectiles oblongs autour de leur centre de gravité. Paris, Gauthier-Villars, 1911. 8vo. 76 pp. Fr. 2.50

SPOFFORD (C. M.). The theory of structures. London, Hill, 1912. 8vo. 17s.

The following papers were read at this meeting:

(1) Mr. S. A. Joffe: "Sums of like powers of natural numbers."

(2) Professor G. A. Miller: "Second note on the groups generated by operators transforming each other into their inverses."

(3) Dr. S. Lefschetz: "On remarkable points of curves."

(4) Dr. S. E. Urner: "Certain singularities of point transformations in space of three dimensions."

(5) Professor A. B. Coble: "The characteristic theory of the odd and even theta functions as related to finite geometry."

(6) Dr. H. H. Mitchell: "Some quaternary groups with particular prime moduli."

(7) Dr. J. E. Rowe: "The undulation and cusp invariants of the R^n."

(8) Professor W. F. Osgood: "A necessary and sufficient condition that a single-valued function in a projective space be rational."

(9) Dr. Dunham Jackson: "On the convergence of the development of a continuous function according to Legendre's polynomials."

(10) Mr. K. P. Williams: "The solutions of non-homogeneous linear difference equations and their asymptotic forms."

(11) Dr. E. J. Miles: "Note on the isoperimetric problem with discontinuous integrand."

(12) Dr. Dunham Jackson: "On approximation by trigonometric sums and polynomials."

(13) Dr. J. E. Hodgson: "Orthocentric properties of the plane directed n-line."

Mr. Williams was introduced by Professor Birkhoff. Dr. Hodgson's paper was communicated to the Society by Professor Morley. In the absence of the authors the papers of Professor Miller, Dr. Lefschetz, Dr. Urner, and Dr. Hodgson were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. By the use of the identity

$$2x(e^{2x} + e^{4x} + \cdots + e^{2nx}) = e^{(2n+1)x} \cdot \frac{2x}{e^{x} - e^{-x}} - \frac{2x}{1 - e^{-2x}}$$

Mr. Joffe expresses the sum of the odd, $(2k - 1)$th (even,

$2k$th) powers of the first n positive integers as an integral function of $2n + 1$ (even, of order $2k$; odd, of order $2k + 1$). Noticing then that any even function of $2n + 1$ is an integral function of $n^2 + n$, he establishes the following theorem: The sum of the odd $(2k - 1)$th powers of the natural numbers $1, 2, \cdots, n$ is an integral function (of order k) of $n^2 + n$, with fractional coefficients. For the sum of even powers there is established a similar theorem, but in this case the integral function of $n^2 + n$ is multiplied by $2n + 1$. The general form of the coefficients in both these cases is rather complicated; a few of the initial and terminal coefficients are simplified and are expressed in terms of Bernoullian numbers. Tables are appended giving the numerical values of the coefficients for all the functions corresponding to the first twenty-five powers.

2. Professor Schur recently called attention to several errors in a brief note by Professor Miller entitled "Groups generated by operators which transform each other into their inverses," [cf. *Jahrbuch über die Fortschritte der Mathematik*, volume 40 (1911), page 189]. In the present note Professor Miller develops the theory of these groups much more completely and observes that the true theorems relating to the points in question are even more general than the false ones which they aim to replace. Among these are the following: There is one and only one group of order 2^n, $n > 2$, which can be generated by a set of m operators such that each of them transforms each of the $m - 1$ remaining ones into its inverse and that no two of them are commutative. When $n = 3$ this is the quaternion group, and when $n = 4$ it is the Hamiltonian group of order 16, but it is not Hamiltonian for any larger value of n. It contains two and only two invariant operators when n is odd, and when n is even it involves exactly four such operators. These constitute the cyclic group of order 4 when n is of the form $4k + 2$ and only then.

All the operators of this group of order 2^n have orders which divide 4, and all these operators of order 4 have a common square. This square generates the commutator subgroup. When n is of the form $4k + 2$ exactly half of the operators of the group are of order 4, since the product of an invariant operator of order 4 and an operator of order 2 or 4 is of order 4 or 2 respectively in the group under consideration. The number of the operators of order 2 for the other values of n can be readily stated in terms of general formulas.

3. Dr. Lefschetz's paper is an attempt to precise some notions relative to remarkable points of curves. Given a plane curve the arc of which is everywhere analytic, and a discrete aggregate of points and lines, curves of a given order m through λ of the points and tangent to μ of the lines will in general have a contact of order at most k with the curve at an arbitrary point. For certain points however the contact will be of order $k + 1$, and these are defined as remarkable points. This definition is also extended to points of contact of multiply tangent curves, and to the similar points obtained by consideration of the reciprocal polar of the system with respect to any conic. It is then shown that, when m, λ, μ vary, the points defined belong to a discrete aggregate on the curve, and that the aggregate of lines of the plane that meet the curve in none of the remarkable points has the power of the continuum.

4. Dr. Urner discusses the behavior of a point transformation at a point where its Jacobian vanishes, attention being given chiefly to the transformation of contact of curves and surfaces. The notion of order of singularity is developed, and criteria are given for its determination.

5. A glance at the formulas for the integer transformation of the periods of the odd and even theta functions of p variables shows that the period and theta characteristics are transformed linearly and homogeneously under a collineation group in the finite domain S_{2p-1}, mod 2, which has an invariant null system. The transformations of the period characteristics contain the coefficients of the integer transformation linearly, i. e., they are transformed like the points (or their null S_{2p-2}'s) of the S_{2p-1}. The transformations on the theta characteristics contain the coefficients to the second degree but not homogeneously. Since $c^2 \equiv c$, mod 2, homogeneity can be effected and the theta characteristics are then transformed like a system of 2^{2p} quadrics in S_{2p-1}—precisely those quadrics whose polar systems coincide with the given null system. The even and odd characteristics correspond to quadrics of different type, containing respectively

$2^{p-1}(2^p+1)-1$ or $2^{p-1}(2^p-1)-1$ points. The object of Professor Coble's paper is to show that the entire theory of the dependence of these characteristics can be regarded as an image of the finite geometry and can be established with great facility from this point of view.

6. In a study of the finite quaternary linear groups with ordinary coefficients, some particular quaternary modular groups were noticed by Dr. Mitchell. A group of order 40,320 with the coefficients of the transformations in the $GF(3^2)$ was found to exist. This was shown to have a self-conjugate subgroup of order 20,160, holoedrically isomorphic with the ternary simple group consisting of all transformations with coefficients in the $GF(2^2)$ and determinant unity.

A group holoedrically isomorphic with the symmetric group on seven letters and with the coefficients of the transformations in the $GF(7)$ was also found to exist.

7. It is known that the invariants of the R^n are expressible in terms of the three-rowed determinants of the matrix of coefficients of the three n-ics in the parametric equations of the R^n. Dr. Rowe's paper consists of a formal statement of a method by means of which the undulation condition can be expressed as a determinant of order $4(n-3)$ and the cusp invariant as one of order $2(n-1)$, the constituents of these determinants being the three-rowed determinants mentioned above. An especially interesting feature is the relation which exists between the cusp and undulation conditions of the R^5 in the plane and the analogues of these two singularities of the R^{2k+1} in space of k dimensions.

8. Professor Osgood shows that a single-valued function of n complex variables, homogeneous and of dimension 0, which is analytic at every finite point of the space of these variables distinct from the point $(0, 0, \cdots)$, or has at most an unessential singularity, is a rational function of these variables.

9. Following investigations of Lebesgue on the subject of Fourier's series (*Bulletin de la Société Mathématique de France*, 1910), Dr. Jackson's paper is devoted to the study of the order of the approximation to a continuous function given by the partial sum, to terms of the nth degree of the expansion

of the function in series of Legendre's polynomials. It is found that, if $f(x)$ is a function such that the difference between two of its values does not exceed $\omega(\delta)$ when the difference of the corresponding values of the argument does not exceed δ, where $\omega(\delta)$ is any function which approaches zero with δ and satisfies one or two other restrictions, then the error of the approximation referred to, in the interior of the interval $(-1, 1)$, does not exceed in magnitude a quantity of the order of $\omega(1/n) \log n$. In particular, if $\omega(\delta) = \text{constant} \times \delta$ (Lipschitz condition), the upper limit for the error so obtained is of the order of $\log n/n$; and if $\lim_{\delta=0} \omega(\delta) \log \delta = 0$ (Lipschitz-Dini condition), this upper limit approaches zero when $n = \infty$, that is, the series converges. An analogous result is obtained when it is assumed that $f(x)$ has a $(k-1)$th derivative satisfying a Lipschitz condition, the corresponding upper limit being of the order of $\log n/n^k$.

It is further demonstrated by an example that the hypothesis made in the first theorem is not sufficient in itself to ensure any more rapid rate of convergence than that which was actually shown to be attained, so that the theorem may to this extent be regarded as satisfactory.

10. In this paper Mr. Williams considers the system of non-homogeneous linear difference equations

$$g_i(x+1) = \sum_{j=1}^{n} a_{ij}(x)\, g_j(x) + b_i(x) \qquad (i = 1, 2, \cdots, n),$$

where the functions $a_{ij}(x)$ and $b_i(x)$ are such that

$$a_{ij}(x) = x^{\mu}\left(a_{ij} + \frac{a_{ij}^{(1)}}{x} + \frac{a_{ij}^{(2)}}{x^2} + \cdots\right),$$

$$b_i(x) = x^{\nu}\left(b_i + \frac{b_i^{(1)}}{x} + \frac{b_i^{(2)}}{x^2} + \cdots\right)$$

$$(i, j = 1, 2, \cdots, n;\ |x| > R).$$

By a direct use of the well known sum formula, obtained by the method of variation of "constants," which has previously been regarded as purely formal, it is shown that in general there exist two simple solutions $g_{11}(x), g_{21}(x), \cdots, g_{n1}(x)$ and $g_{12}(x), g_{22}(x), \cdots, g_{n2}(x)$.

The first solution is analytic throughout the finite plane, save for possible singularities at the singularities of the functions $a_{ij}(x)$, $b_i(x)$, and the zeros of the determinant* $|a_{ij}(x)|$, and points any number of units to the left of these points; the second solution is analytic except at points any number of units to the right of the singularities of $a_{ij}(x)$, $b_i(x)$. These two solutions are shown to be asymptotic, in the right and left half planes respectively, to the series formally satisfying the given system; and one of them maintains its asymptotic form in the remaining half plane at a sufficient distance from the axis of reals.

The method employed to make the sum formula yield these solutions is to replace the sums in one case by an infinite series, and in the other by an appropriate contour integral.

Like results are derived for a single non-homogeneous linear difference equation of the nth order, reducible to such a system.

11. In dealing with the isoperimetric problem of the calculus of variations, where one is asked to find and discuss the properties of curves

$$D: \quad x = \varphi(t), \quad y = \psi(t)$$

which minimize the integral

$$J \equiv \int F(x, y, x', y')dt$$

subject to the condition

$$K \equiv \int G(x, y, x', y')dt = L,$$

it is customary to assume that the functions F and G are continuous in their four arguments. In Dr. Miles's note some properties of the minimizing curve D are given when the functions F and G are allowed to have a finite discontinuity as the point (x, y) passes through a given curve M.

12. In his thesis (Göttingen, 1911) Dr. Jackson has proved the following theorem: If $f(x)$ is a function which possesses a derivative of order $k-1$ satisfying a Lipschitz condition throughout a closed interval, then $f(x)$ may be approximately represented in the interval by a polynomial of degree n or lower, with a maximum error not exceeding K/n^k, where k is independent of n. The most interesting special case is that in which $k = 1$ and $f(x)$ itself satisfies a Lipschitz condition. On the assumption that $f(x)$ has the period 2π and

satisfies the conditions stated above for all values of x, it was shown that a precisely analogous theorem holds for the approximation of $f(x)$ by a trigonometric sum of order n or lower, this result being obtainable as a consequence of the preceding. It is now shown that decided simplification in the proof of both theorems may be effected by proving the second directly (this had been done only for $k = 1$) and deducing the first from it.

This method has the further advantage that the numerical constants involved can be computed more conveniently. For example, if $f(x)$ satisfies the condition

$$|f(x_2) - f(x_1)| \leqq |x_2 - x_1|$$

in the closed interval (0, 1), it can be approximately represented in this interval by a polynomial of degree n or lower, with an error which never exceeds $3/n$, for all positive integral values of n. The same line of investigation leads to results in the theory of Fourier's series.

13. There is a theorem that the perpendiculars let fall from the incenters of three out of four lines of given direction upon the remaining line touch a circle. In Dr. Hodgson's paper a circle is obtained for any even number of lines, beginning with four. If we take this circle for any $2n$ out of $2n + 1$ lines, the $2n + 1$ circles touch a line. The question of the reversal of direction of one or more of $2n$ lines is then taken up, and this is followed by the consideration of the configuration of circles arising from four, five, and six lines.

F. N. COLE,
Secretary.

ON THE FOUNDATIONS OF THE THEORY OF LINEAR INTEGRAL EQUATIONS.*

BY PROFESSOR E. H. MOORE.

1. *The Analogous Systems of Linear Equations.*

THE theory of linear integral equations, mathematically considered, has its taproot in the classical analogies between an algebraic sum, the sum of an infinite series, and a definite integral.

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*Address of the Vice-President and Chairman of Section A of the American Association for the Advancement of Science, Washington, December 29, 1911.

Consider the linear algebraic equation

$$(\mathrm{I}^\circ) \qquad x = ky$$

for the number y, the coefficient k and the number x being given. From this single equation I° we ascend to the algebraic system

$$(\mathrm{II}_n^\circ) \qquad x_i = \sum_{j=1}^{n} k_{ij} y_j \qquad (i = 1, 2, \cdots, n)$$

of n simultaneous linear equations for the determination of the set (y_i) of n numbers $y_1, \cdots, y_n$, the matrix (k_{ij}) of n^2 coefficients $k_{11}, \cdots, k_{nn}$ and the set (x_i) of n numbers $x_1, \cdots, x_n$ being given.

To this algebraic system (II_n°) we have by the classic analogy the two corresponding transcendental systems

$$(\mathrm{III}^\circ) \qquad x_i = \sum_{j=1}^{\infty} k_{ij} y_j \qquad (i = 1, 2, \cdots);$$

$$(\mathrm{IV}^\circ) \qquad \xi(s) = \int_a^b \kappa(s, t)\eta(t)dt \qquad (a \leqq s \leqq b).$$

In III° the infinite set (y_i) is to be found, the infinite set (x_i) and the infinite matrix (k_{ij}) of coefficients being given; the suffixes i, j have the range $1, 2, \cdots$. In IV° the unknown function η and the known function ξ are functions of one variable ranging over the interval a-b of the real number system, while the known coefficient function or, in Hilbert's terminology, kernel κ is a function of two variables ranging independently over that interval. It is plain that the theories of III°, IV° must involve convergence considerations. Throughout, the numbers and the functional values of the functions are real or complex numbers.

You are aware that the study of the algebraic system II_n°, initiated before 1678 by the genial intuition of the philosopher-mathematician Leibniz, led to the development of the theory of determinants—a theory which in the nineteenth century came to permeate all branches of number theory, algebra, analytic geometry, and pure and applied analysis, exerting everywhere a profound influence not merely by its usefulness but perhaps even more by the extreme elegance of its methods and results.

The theory of infinite determinants connected with the denumerably infinite system III°, or more exactly with the equivalent system in which the indices i, j have the integral values from $-\infty$ to $+\infty$, was initiated by G. W. Hill, who in 1877 made hardy but happily effective use of the determinant of a system of the latter type in his solution of a differential equation arising in his memorable study of the motion of the lunar perigee. To supply the requisite convergence proofs, Henri Poincaré in 1886 laid the foundations of the general theory of infinite determinants, which has since been developed chiefly by Helge von Koch.

We are led to an analogous determinant, not from the continuously infinite system IV° of linear equations, but from the system

$$\text{(IV)} \qquad \xi(s) = \eta(s) - z\int_a^b \kappa(s, t)\eta(t)dt \quad (a \leqq s \leqq b).$$

Here z is a given number, real or complex, and we consider the *regular* case, in which the functions involved are continuous real or complex valued functions of their arguments.

The corresponding systems

$$\text{(I)} \qquad x = y - zky,$$

$$\text{(II}_n\text{)} \qquad x_i = y_i - z\sum_{j=1}^{n} k_{ij}y_j \quad (i = 1, 2, \cdots, n),$$

$$\text{(III)} \qquad x_i = y_i - z\sum_{j=1}^{\infty} k_{ij}y_j \quad (i = 1, 2, \cdots)$$

are respectively equivalent to the systems I°, II$_n$°, III°.

The types IV, IV° are however essentially distinct. For instance, if we look at IV and IV° as transformations of the functions η into the functions ξ, the type IV contains (in the case of vanishing parameter z or identically vanishing coefficient function or kernel κ) the identical transformation $\xi = \eta$, while the type IV° does not contain this transformation.

The solution η of the system or integral equation IV may be expanded formally as a power series in the parameter z, viz.,

$$\eta(p) = \xi(p) + z\int_a^b \kappa(p, p_1)\xi(p_1)dp_1 + z^2\int_a^b\int_a^b \kappa(p, p_1)\kappa(p_1, p_2)\xi(p_2)dp_1dp_2 + \cdots.$$

This power series converges near $z = 0$ uniformly in p, and accordingly for z sufficiently small has as sum a continuous function η which is readily proved to be a solution of IV.

Impelled by the fact that integral equations of type IV occur very frequently in the linear problems of mathematical physics, in the late nineties of the last century the Swedish mathematical physicist Ivar Fredholm undertook the study of the analytic character of the solution η as a function of the parameter z. After earlier notes on the subject, Fredholm published his fundamental memoir in 1903 in volume 27 of the *Acta Mathematica*, in one of the two volumes of the *Acta* dedicated to the memory of Abel on the occasion of the centenary of his birth. And this was the more fitting since Abel first studied special integral equations of the type IV°. Accordingly Fredholm calls the equation IV° Abel's integral equation. Mathematicians generally call the equation IV Fredholm's integral equation. With Hilbert one also designates the equations IV°, IV as integral equations of the first and second kind.

Fredholm found that the function η is a single-valued analytic function of the parameter z, having at most polar singularities in the finite z-plane, and he exhibited it explicitly as the quotient of two permanently converging power series in z. The denominator series with coefficients depending only on the kernel κ, viz.,

$$1 - z\int_a^b \kappa(p, p)\,dp + \frac{z^2}{2}\int_a^b\int_a^b \begin{vmatrix} \kappa(p_1, p_1) & \kappa(p_1, p_2) \\ \kappa(p_2, p_1) & \kappa(p_2, p_2) \end{vmatrix} dp_1 dp_2 \pm \cdots,$$

is Fredholm's determinant of the kernel κ with parameter z. In case z is not a root of this determinant, for every function ξ there is a definite solution η of the equation IV, and the same is true as to the adjoint equation

$$\xi(t) = \eta(t) - z\int_a^b \eta(s)\kappa(s, t)\,ds \quad (a \leqq t \leqq b). \tag{$\breve{\text{IV}}$}$$

On the other hand, if z is a root of the determinant, it is of finite multiplicity m, and each of the corresponding homogeneous equations

$$\eta(s) = z\int_a^b \kappa(s, t)\eta(t)\,dt \qquad (a \leqq s \leqq b), \tag{IV_H}$$

$$\eta(t) = z\int_a^b \eta(s)\kappa(s, t)\,ds \qquad (a \leqq t \leqq b) \tag{$\breve{\text{IV}}_H$}$$

has a solution η not identically vanishing, the number n of linearly independent solutions η for one equation being the same as for the other equation and at most m.

These few results suffice to suggest the close parallelism between Fredholm's theory of the integral equation IV and the current theory of the algebraic system II_n.

We have seen, then, that the theories of determinants of the matrices or kernels of the three types—the finite, the denumerably infinite, the continuously infinite—were initiated by the mathematician-philosopher Leibniz, the mathematical astronomer Hill, the mathematical physicist Fredholm; and we appreciate anew the magnitude of the debt owed by pure mathematics to its most closely related sister sciences—logic, astronomy, physics—a debt abundantly repaid by the applications throughout the wide range of the sciences, at least in the progress of time, of even the most abstract doctrines of pure mathematics.

2. *References to the Literature.*

The investigations of von Koch and Fredholm opened the way for the systematic development, now in rapid progress, of the analogies and the interrelations between the algebraic and the two kinds of transcendental theories, and for the immediate application of the new results in various domains of pure and applied analysis.

Especially noteworthy are the memoirs of David Hilbert and of Erhard Schmidt. By direct limiting processes Hilbert obtains from algebraic theorems the corresponding transcendental theorems. Hilbert has thus initiated a theory of functions of a denumerable infinity of variables, from which in turn, by the connection between functions of continuous variables and their Fourier coefficients, one proceeds to the theory of functions of continuous variables. In particular, Hilbert has studied real-valued symmetric kernels, obtaining the transcendental analogues, for example, of the orthogonal transformation of the algebraic quadratic form to the sum of squares of linear forms. Geometric analogies of metrical nature play a considerable rôle in the work of Hilbert, and perhaps even more in the work of Schmidt, who treats transcendental problems directly, using methods originated by H. A. Schwarz in the potential theory. Schmidt has also entered upon the study of non-linear integral equations.

But for details of the extensive literature and present state of the whole subject I must content myself with referring to the most recent books:

HEYWOOD ET FRÉCHET: L'équation de Fredholm et ses applications à la physique mathématique. Hermann, Paris, 1912.

LALESCO: Introduction à la théorie des équations intégrales. Hermann, Paris, 1912.

and to the reports:

HILBERT: "Wesen und Ziele einer Analysis der unendlichvielen unabhängigen Variabeln." *Palermo Rendiconti*, volume 27, pages 59–74 (1909).

VON KOCH: "Sur les systèmes d'une infinité d'équations linéaires à une infinité d'inconnues." *Compte rendu du Congrès des Mathématiciens*, tenu à Stockholm 22–25 Septembre, 1909, pages 43–61; Teubner, Leipzig, 1910.

FREDHOLM: "Les équations intégrales linéaires." Ibid., pages 92–100.

BATEMAN: "Report on the history and present state of the theory of integral equations." Report to the British Association for the Advancement of Science, Sheffield, 1910, 80 pp.; Burlington House, London, 1911.

HAHN: "Bericht über die Theorie der linearen Integralgleichungen." *Jahresbericht der Deutschen Mathematiker-Vereinigung*, volume 20, pages 69–117 (1911).

3. *The Fundamental Problem of Unification. General Analysis.*

We are now in position to take up, as the specific subject of this discourse, the question of foundations of the theory of linear integral equations. We have seen that the algebraic theory serves to suggest the corresponding transcendental theories, or even to determine those theories by suitable use of limiting processes. But this state of affairs may be recognized as only preliminary to the determination of a general theory capable of specialization into the various theories. This is in accordance with a general heuristic principle of scientific procedure, which I have formulated as follows:

The existence of analogies between central features of various theories implies the existence of a general theory which underlies the particular theories and unifies them with respect to those central features.

For the case of the real-valued symmetric kernel, and in fact for the more general case of the complex-valued hermitian kernel, i. e., a kernel κ satisfying identically the condition that $\kappa(s, t)$ and $\kappa(t, s)$ are conjugate complex numbers, I took up six years ago this problem of unification for the Hilbert theory as presented by Schmidt. This was the theme of my series of lectures: "On the theory of bilinear functional operations," at the colloquium of the American Mathematical Society, held in September, 1906, in New Haven, under the auspices of Yale University.

Subsequent study led to the recognition that the general theory of linear integral equations is merely a division in the theory of a certain form of general analysis, an introduction to which, instead of the colloquium lectures, I published,[*] as pages 1–150, in the volume The New Haven Mathematical Colloquium, etc., Yale University Press, New Haven, 1910.

This morning I wish to establish, in the sense of general analysis, an adequate and satisfactorily simple foundation for the general theory of linear integral equations, embracing by specialization, as we shall see, together with an interesting variety of new theories, the algebraic and both types of transcendental theories, now current, at least in so far as regular kernels are concerned.

As to the system III, I call the matrix-kernel (k_{ij}) regular in case there is a set (k_i) of numbers of finite norm $\Sigma_i |k_i|^2$, such that for every i and j $|k_{ij}| \leqq |k_i k_j|$. This regular kernel satisfies the latest condition found[†] by von Koch as sufficient that the infinite determinant and all its minors converge absolutely. Then, if the sets (x_i), (y_i) are of finite norm, the system III may be treated either by the method of infinite determinants or by Hilbert's theory of functions of infinitely many variables.

I shall indicate, first, the terminology or *basis* of the foundation of the general theory, and then two sets of *postulates*, the former effective for the validation of the general Fredholm theory for the general kernel, and the latter effective for the validation of the general Hilbert-Schmidt theory of the real symmetric or the more general hermitian kernel. And, in advance, I notice that for the latter theory we need more postulates than for the former theory. This is in accordance with the nature of things logical: *we must pay for the elaboration of theory by the imposition of additional postulates and the corresponding restriction of scope of application.*

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[*] Cf. also my paper, "On a form of General Analysis, with application to linear differential and integral equations," read before the Section on Analysis of the Rome Congress of 1908, Atti, etc., vol. 2 (1909), pp. 98–114.

[†] Loc. cit., pp. 49, 50; *Palermo Rendiconti*, vol. 28 (1909), pp. 257, 263. The condition is that the kernel (k_{ij}) have the form $k_{ij} = u_{ij} v_i / v_j \quad (i, j)$, where $v_i \neq 0 \quad (i)$ and the series $\Sigma_{ij} u_{ij}^2$, $\Sigma_i u_{ii}$ converge absolutely.

4. *Fredholm's Equation in General Analysis. The Basis* Σ_1.

In order to bring the equation-systems I, II, III into notational conformity with the integral equation IV, we regard, for instance, in III the set (x_i) of numbers x_i $(i = 1, 2, 3, \cdots)$ as a function x or ξ of the argument i or s with the range $i = s = 1, 2, 3, \cdots$. Then I, II$_n$, III, IV are special cases of the general equation

$$(G) \qquad \xi = \eta - zJ\kappa\eta,$$

with the meaning

$$(G) \qquad \xi(s) = \eta(s) - zJ_t\kappa(st)\eta(t) \quad (s),$$

which we designate as Fredholm's equation in general analysis. The kernel κ, the parameter z, and the function ξ being given, the function η is to be determined as a solution of the equation G.

The understanding is that (1) ξ and η are functions of an argument p or s or t having a certain range $\mathfrak{P}$; (2) κ or $\kappa(st)$ is a function of two arguments ranging independently over $\mathfrak{P}$; (3) J or J_t is a functional operation turning a product $\kappa\eta$ or $\kappa(st)\eta(t)$ into a function of the argument s; and (4) the equation G holds for every value of s on the range $\mathfrak{P}$.

For the general theory this range $\mathfrak{P}$ is simply a class of elements p. These elements p are of any nature whatever, e. g., numbers, sets of numbers, functions, points, curves; and they are not necessarily all of the same nature. Thus, the range $\mathfrak{P}$ is a general class of general elements. This "general" is the *true general,* in the sense of *arbitrarily special,* that is, capable of arbitrary specification—*without the exclusion* of exceptional or singular cases.

Thus, for the general theory of the equation G the range $\mathfrak{P}$ enters without the imposition of restrictive properties or features, and it is *this presence in the theory of a general class* which *constitutes the theory a doctrine of the form of general analysis* which we are developing.

For the respective instances II$_n$, III, IV, the range $\mathfrak{P}$ is *finite, denumerably infinite, continuously infinite,* consisting of the respective elements

$$p = 1, 2, \cdots, n; \quad p = 1, 2, 3, \cdots; \quad a \leqq p \leqq b.$$

We denote these ranges by the notations

$$\mathfrak{P}^{\mathrm{II}_n}; \quad \mathfrak{P}^{\mathrm{III}}; \quad \mathfrak{P}^{\mathrm{IV}}.$$

The respective functional operations J are

$$J_t^{\mathrm{II}_n} \equiv \sum_{t=1}^{n}; \quad J_t^{\mathrm{III}} \equiv \sum_{t=1}^{\infty}; \quad J_t^{\mathrm{IV}} \equiv \int_a^b dt .$$

In the instances III and IV the functions ξ, η, κ are necessarily subject to certain conditions of convergence or of continuity. The conditioning properties are defined in terms of features possessed by the special classes $\mathfrak{P}^{\mathrm{III}}$, $\mathfrak{P}^{\mathrm{IV}}$; we are able to speak in $\mathfrak{P}^{\mathrm{III}}$ of p tending to ∞, and in $\mathfrak{P}^{\mathrm{IV}}$ of the difference $p_1 - p_2$ of two elements p. Similarly, in the general theory the functions must possess certain properties, which must however be postulated and not explicitly defined, since we attribute to the general range no features available for use in the definitions. Now, instead of postulating *properties of the functions,* it is technically more convenient to postulate *classes of functions* to which the functions shall belong, viz., the classes of functions are the classes of all functions possessing the respective properties.

Accordingly, the form of the general equation G leads to the following first basis:

$$\Sigma_1 = (\mathfrak{A};\ \mathfrak{P};\ \mathfrak{M};\ \mathfrak{K};\ J)$$

for the construction of a general theory of the linear equation G. Here $\mathfrak{A}$ denotes the class of all real or the class of all complex numbers a; $\mathfrak{B}$ denotes a general class of general elements p or s or t or u or v or w; $\mathfrak{M}$ denotes a class of single-valued functions μ on $\mathfrak{P}$ to $\mathfrak{A}$, that is, for every function μ and argument p, $\mu(p)$ denotes a definite number a of the class $\mathfrak{A}$; $\mathfrak{K}$ denotes a class of functions κ on $\mathfrak{P}\mathfrak{P}$ to $\mathfrak{A}$, that is, for every function κ and (ordered) pair (st) of arguments of $\mathfrak{P}$, $\kappa(st)$ denotes a definite number a of the class $\mathfrak{A}$; and J or J_t denotes a definite functional operation on $\mathfrak{K}\mathfrak{M}$ or $\mathfrak{K}_{st}\mathfrak{M}_t$ to $\mathfrak{M}$ or $\mathfrak{M}_s$, that is, for every function κ of $\mathfrak{K}$ and η of $\mathfrak{M}$, $J\kappa\eta$ or $J_t\,\kappa(st)\eta(t)$ denotes a definite function of $\mathfrak{M}$ or $\mathfrak{M}_s$.

For the basis Σ_1 *the problem of foundations* of a general theory of the equation G is then: *to specify properties of the constituents of the basis* Σ_1 *sufficient to validate the desired theory.* The range $\mathfrak{P}$ is to remain general, and the properties specified are to be *of general reference,* that is, defined with respect to the general range $\mathfrak{P}$.

It is convenient here to refer to the important memoirs of S. Pincherle:

"Sulle equazioni funzionali lineari". *Rendiconti della R. Accademia dei Lincei,* ser. 5, vol. 14 (1905), pp. 366–374;

"Sulle equazioni funzionali lineari". *Bologna Memorie,* ser. 6, vol. 3 (1906), pp. 143–171;

"Appunti di calcolo funzionali". *Bologna Memorie,* ser. 6, vol. 8 (1911), pp. 1–38.

In these memoirs, from the standpoint of the general theory of linear distributive functional operations, Pincherle investigates the problem of foundations for a theory of the equation

$$(G') \qquad \xi = \eta - zJ'\eta,$$

which includes the equation G, with the basis

$$\Sigma' = (\mathfrak{A};\ \mathfrak{P};\ \mathfrak{M};\ J'),$$

where J' is a functional operation on $\mathfrak{M}$ to $\mathfrak{M}$.

5. *Certain Definitions. The Closure Property C_1. Relative Uniformity of Convergence.*

We do not however retain the basis Σ_1. With the purpose of obtaining finally a general theory characterized by its simplicity and by its possession of a number of important *closure properties,* we set up other bases $\Sigma_2, \cdots, \Sigma_6$. Whenever a general theory, as a matter of fact, includes, as a special instance, a theory analogous to but not *a priori* one of its instances, we speak of a closure property of the general theory. The terminology is adapted from that in use in the theory of point sets.

That the general theory include, as special instances, the current theories of the equations I, II_n, III, IV with regular kernels is the closure property C_1, fundamental to the whole inquiry.

You are familiar with the notion, *uniformity of convergence* of a sequence of functions over a range of values of the argument of the functions, and you appreciate the fundamental rôle played by the notion in the development of analysis during the last half century. A sequence $\{\mu_n\}$ of functions μ_n $(n = 1, 2, 3, \cdots)$ of the argument p on the range $\mathfrak{P}$ converges over the range $\mathfrak{P}$ uniformly to a function θ as limit function, in notation

$$\underset{n}{L}\, \mu_n = \theta \quad (\mathfrak{P}),$$

in case for every positive number e and index n greater than a number n_e, dependent upon e alone, the difference $\mu_n(p) - \theta(p)$ is, for every value of p on the range $\mathfrak{P}$, in absolute value at most e:

$$| \mu_n(p) - \theta(p) | \leqq e.$$

For investigations in general analysis we need a more general notion, the notion of *relative uniformity of convergence*, or, *uniformity of convergence relative to a scale function.* If the function σ (defined on the same range $\mathfrak{P}$) is the scale function, this relative uniformity, in notation

$$\underset{n}{L}\, \mu_n = \theta \quad (\mathfrak{P};\ \sigma),$$

has the same definition except that the final inequality is replaced by

$$| \mu_n(p) - \theta(p) | \leqq e\ |\sigma(p)|.$$

We speak also of relative uniformity as to a class $\mathfrak{S}$ of scale functions; the notation

$$\underset{n}{L}\, \mu_n = \theta \quad (\mathfrak{P};\ \mathfrak{S})$$

means that for some function σ of the class $\mathfrak{S}$ we have the relative uniformity as to the scale function σ.

One observes that the classical uniformity is that instance of relative uniformity in which the scale function is identically 1.

In illustration of this notion of relative uniformity, consider a sequence $\{\mu_n\}$ of real-valued nowhere negative functions μ_n of the real variable p on the infinite interval $\mathfrak{P} = 1 - \infty$ of the number system, and let the functions μ_n be individually integrable from 1 to ∞. If relative to a scale function σ of the same kind the sequence $\{\mu_n\}$ converges uniformly on $\mathfrak{P}$ to a limit function θ, then θ also is integrable from 1 to ∞, and the integral of θ is the limit as to n of the integral of μ_n. But the limit function θ is no longer necessarily integrable from 1 to ∞, if the convergence is merely uniform in the classical sense ($\sigma = 1$), as appears from the following example:

$$\mu_n(p) = 1/p \quad (1 \leqq p \leqq n), \quad n/p^2 \ (p \geqq n),$$
$$\theta(p) = 1/p \qquad (1 \leqq p).$$

To facilitate the exposition of the sequel we need certain additional definitions.

Consider a class $\mathfrak{M}$ of functions μ on $\mathfrak{P}$ to $\mathfrak{A}$, that is, on the range $\mathfrak{P}$ with functional values belonging to the class $\mathfrak{A}$. The class $\mathfrak{M}_L$, *the linear extension of the class* $\mathfrak{M}$, is the class of all functions μ_L of the form

$$\mu_L = a_1\mu_1 + a_2\mu_2 + \cdots + a_n\mu_n,$$

viz., the class of all linear combinations of a finite number of functions belonging to the class $\mathfrak{M}$ with numerical coefficients belonging to the class $\mathfrak{A}$. Further, $\mathfrak{S}$ being a class of functions σ on $\mathfrak{P}$ to $\mathfrak{A}$, the class $\mathfrak{M}_{\mathfrak{S}}$, *the class* $\mathfrak{M}$ *extended as to the class* $\mathfrak{S}$, is the class of all functions θ of the form

$$\theta = \underset{n}{L}\, \mu_n \quad (\mathfrak{P};\ \sigma),$$

a form which has been defined above. The class $\mathfrak{M}$ is contained in the class $\mathfrak{M}_L$ and for every $\mathfrak{S}$ in the class $\mathfrak{M}_{\mathfrak{S}}$. If the classes $\mathfrak{M}$ and $\mathfrak{M}_L$ are identical, the class $\mathfrak{M}$ is said to be *linear*, to have the property L. If the classes $\mathfrak{M}$ and $\mathfrak{M}_{\mathfrak{M}}$ are identical, the class $\mathfrak{M}$ is said to be *closed*, to have the property C; otherwise expressed, the class $\mathfrak{M}$ is closed in case every function θ of the form

$$\theta = \underset{n}{L}\, \mu_n \quad (\mathfrak{P};\ \mu)$$

belongs to the class $\mathfrak{M}$. The class $\mathfrak{M}_*$, *the *-extension of the class* $\mathfrak{M}$, is the class $(\mathfrak{M}_L)_{\mathfrak{M}}$, that is, the extension as to $\mathfrak{M}$ of the linear extension of $\mathfrak{M}$. The class $\mathfrak{M}^2$ is the class of all functions of the form $\mu_1\mu_2$ or $\mu_1(p)\mu_2(p)$, that is, of all products of pairs of functions of the class $\mathfrak{M}$, the arguments of the two functions being the same.

In illustration of these definitions, if $\mathfrak{M}$ is the class $\mathfrak{M}^{\mathrm{IV}}$ of all continuous functions on the finite linear interval $\mathfrak{P}^{\mathrm{IV}}$, a-b, we have

$$\mathfrak{M} = \mathfrak{M}_L = \mathfrak{M}_{\mathfrak{M}} = \mathfrak{M}_* = \mathfrak{M}^2.$$

Further, if $\mathfrak{M}$ is the class $\mathfrak{M}^{\mathrm{III}}$ of all functions μ on $\mathfrak{P}^{\mathrm{III}}$

$(p = 1, 2, 3, \cdots)$ such that the series $\Sigma_p \mu(p)^2$ converges absolutely, we have

$$\mathfrak{M} = \mathfrak{M}_L = \mathfrak{M}_{\mathfrak{M}} = \mathfrak{M}_*; \quad \mathfrak{M}^2 = \mathfrak{M}^{\mathrm{III}_1},$$

where $\mathfrak{M}^{\mathrm{III}_1}$ denotes the class of all functions μ on $\mathfrak{P}$ such that the series $\Sigma_p \mu(p)$ converges absolutely. Thus the classes $\mathfrak{M}^{\mathrm{III}_2}$ and $\mathfrak{M}^{\mathrm{IV}}$ occurring in the regular cases of equations III and IV are linear closed classes of functions.

Consider further two general ranges $\mathfrak{P}'$, $\mathfrak{P}''$ conceptually but not necessarily actually distinct. The product range $\mathfrak{P}'\mathfrak{P}''$ is the class of all composite elements (p', p'') or $p'p''$, the first constituent p' being an element p' of the class $\mathfrak{P}'$, and the second constituent p'' being an element p'' of the class $\mathfrak{P}''$. For example, if $\mathfrak{P}'$ is the linear interval a'-b' and $\mathfrak{P}''$ is the linear interval a''-b'', the product $\mathfrak{P}'\mathfrak{P}''$ is, to speak geometrically, the rectangle $a' \leqq p' \leqq b'$, $a'' \leqq p'' \leqq b''$. The *product class* $\mathfrak{M}'\mathfrak{M}''$ of two classes $\mathfrak{M}'$, $\mathfrak{M}''$ of functions on the respective ranges $\mathfrak{P}'$, $\mathfrak{P}''$ consists of all products $\mu'\mu''$ or $\mu'(p')\mu''(p'')$ of a function μ' of the class $\mathfrak{M}'$ on $\mathfrak{P}'$ and a function μ'' of the class $\mathfrak{M}''$ on $\mathfrak{P}''$. The class $(\mathfrak{M}'\mathfrak{M}'')_*$, *the* $*$*-composite of two classes* $\mathfrak{M}'$, $\mathfrak{M}''$ on the respective ranges $\mathfrak{P}'$, $\mathfrak{P}''$ is, as indicated by the notation, the $*$-extension of the product class $\mathfrak{M}'\mathfrak{M}''$, viz., the extension as to the product class $\mathfrak{M}'\mathfrak{M}''$ of the linear extension of the product class $\mathfrak{M}'\mathfrak{M}''$. The classes $\mathfrak{M}'\mathfrak{M}''$ and $(\mathfrak{M}'\mathfrak{M}'')_*$ are classes of functions on the product range $\mathfrak{P}'\mathfrak{P}''$, and accordingly, if the ranges $\mathfrak{P}'$, $\mathfrak{P}''$ are identical, $\mathfrak{P}' = \mathfrak{P}'' \equiv \mathfrak{P}$, the arguments of the functions of those classes are variables $(p', p''$ or $p_1, p_2)$ ranging independently over $\mathfrak{P}$.

The suitability of these notions for use in a general theory of linear integral equations is indicated by the fact that for the regular case of the equation III or IV the functions ξ and η belong to a class $\mathfrak{M}$ ($\mathfrak{M}^{\mathrm{III}_2}$ or $\mathfrak{M}^{\mathrm{IV}}$) whose $*$-composite with itself is the class $\mathfrak{K}$ to which the kernels belong, viz.,

$$\mathfrak{K} = (\mathfrak{M}\mathfrak{M})_*.$$

6. *The Bases* Σ_2, Σ_3, Σ_4.

With the aid of the notions and notations now at hand we are able to proceed rapidly towards our goal. We recall that the basis Σ_1

$$\Sigma_1 = (\mathfrak{A}; \mathfrak{P}; \mathfrak{M}; \mathfrak{K}; J),$$

was dictated by the mere form of the general equation G

$$(G) \qquad \xi(s) = \eta(s) - zJ_t\,\kappa(st)\eta(t) \quad (s).$$

By the consideration that in the typical instances I, II_n, III, IV the kernel $\kappa(st)$ for every s as a function of the argument t belongs to the class $\mathfrak{M}_t$ to which the function $\eta(t)$ belongs, we are led to a basis Σ_2, simpler than Σ_1, in the form

$$\Sigma_2 = (\mathfrak{A};\ \mathfrak{P};\ \mathfrak{M};\ \mathfrak{N} \equiv \mathfrak{M}^2;\ \mathfrak{K};\ J).$$

Here the class $\mathfrak{N}$ is defined as the class $\mathfrak{M}^2$ of all products $\mu_1\mu_2$ of pairs of functions of the class $\mathfrak{M}$, the arguments being the same for the two functions; and J is a functional operation on $\mathfrak{N}$ to $\mathfrak{A}$, so that for every function ν of $\mathfrak{N}$, $J\nu$ denotes a number a of the class $\mathfrak{A}$.

This system Σ_2 was basal for my lectures at the New Haven Colloquium of 1906. The development of the theory of the general equation G on the basis Σ_2 requires numerous postulates. We must, for instance, arrange to extend the scope of the functional operation J from the class $\mathfrak{N}$ to its linear and $*$-extensions, $\mathfrak{N}_L$ and $\mathfrak{N}_*$, in such wise that from the equations

$$\nu_L = a_1\nu_1 + \cdots + a_n\nu_n; \quad \theta = \underset{*}{L}\,\nu_{Ln} \quad (\mathfrak{P};\ \nu)$$

we have the conclusions

$$J\nu_L = a_1J\nu_1 + \cdots + a_nJ\nu_n; \quad J\theta = \underset{*}{L}\,J\nu_{Ln}.$$

Accordingly, if we define the class $\mathfrak{N}$ as the class $\mathfrak{M}_*$, the $*$-extension of the class $\mathfrak{M}$, and take J as a functional operation on this class $\mathfrak{N}$ to $\mathfrak{A}$, we have a simpler basis Σ_3

$$\Sigma_3 = (\mathfrak{A};\ \mathfrak{P};\ \mathfrak{M};\ \mathfrak{N} \equiv \mathfrak{M}_*;\ \mathfrak{K};\ J).$$

A theory based on Σ_3 requires postulates involving the kernel class $\mathfrak{K}$. These postulates are avoided if we define the class $\mathfrak{K}$ as the $*$-composite of the class $\mathfrak{M}$ with itself. Thus we obtain the still simpler basis Σ_4

$$\Sigma_4 = (\mathfrak{A};\ \mathfrak{P};\ \mathfrak{M};\ \mathfrak{N} \equiv \mathfrak{M}_*;\ \mathfrak{K} \equiv (\mathfrak{M}\mathfrak{M})_*;\ J).$$

Here J is still a functional operation on $\mathfrak{N}$ to $\mathfrak{A}$. For the regular cases of the typical instances I, II_n, III, IV the definition just suggested of the kernel class $\mathfrak{K}$ is appropriate.

As to irregular cases, it is clear that a greater variety could be treated on the basis Σ_2 or Σ_3 than on the basis Σ_4.

7. *The Closure Property C_2. The Basis Σ_5.*

In response to the desire that our theory of the general equation G

$$(G) \qquad \xi(s) = \eta(s) - zJ_t\,\kappa(st)\eta(t) \quad (s)$$

shall possess a certain closure property C_2, in addition to the fundamental property C_1 of having as special instances the current theories for the regular cases of the equations (I, II$_n$, III, IV), we are led to the basis Σ_5.

Under the postulates to be imposed on the class $\mathfrak{M}$, the kernel $\kappa(st)$ of the class $\mathfrak{K} \equiv (\mathfrak{M}\mathfrak{M})_*$ is, as in the typical instances, for every s as a function of t of the class $\mathfrak{M}_t$ to which also $\eta(t)$ belongs: thus, the operation J enters the equation G in the form

$$J\alpha\beta \quad \text{or} \quad J_p\,\alpha(p)\beta(p),$$

where α and β are functions of the class $\mathfrak{M}$. Now for the instance II$_n$ the expression

$$J^{\mathrm{II}_n}\alpha\beta \quad \text{or} \quad \sum_{p=1}^{n}\alpha(p)\beta(p)$$

is Grassmann's inner product of the n-dimensional vectors

$$\alpha = (\alpha(1), \cdots, \alpha(n)); \quad \beta = (\beta(1), \cdots, \beta(n)),$$

and accordingly we term the expression $J_p\,\alpha(p)\beta(p)$ or $J\alpha\beta$ the *inner product* of the functions α, β. In this we follow usage already established for the instances III, IV.

For the basis Σ_4 it is convenient temporarily to denote the operation J by J^4 and the equation G by G^4. J^4 is then a functional operation on $\mathfrak{N} \equiv \mathfrak{M}_*^2$ to $\mathfrak{A}$. The first question that suggests itself is the following: Is it possible to secure a basis Σ_5 for the equation G^5

$$(G^5) \qquad \xi = \eta - zJ^5\kappa\eta$$

of such a nature that the corresponding inner product

$$J^5\alpha\beta$$

shall have as one of its instances the generalized inner product

$$J^4{}_t J^4{}_u \, \alpha(t)\omega(tu)\beta(u),$$

where ω is a function of the class $\mathfrak{K}$? To that end we set

$$J^5\alpha\beta \equiv J^5_{(tu)}\alpha(t)\beta(u),$$

and assign to G^5 the meaning

$$(G^5) \qquad \xi(s) = \eta(s) - zJ^5_{(tu)}\kappa(st)\eta(u) \quad (s).$$

Further, since the operand $\alpha(t)\beta(u)$ belongs to the product class $\mathfrak{M}\mathfrak{M}$, just as we were led to replace J^2 on $\mathfrak{M}^2$ by J^3 and J^4 on $\mathfrak{N} \equiv \mathfrak{M}^2_*$, we stipulate that J^5 shall be a functional operation on the class $(\mathfrak{M}\mathfrak{M})_*$, that is, on the kernel class $\mathfrak{K}$. Thus we have reached the basis

$$\Sigma_5 = (\mathfrak{A};\ \mathfrak{P};\ \mathfrak{M};\ \mathfrak{K} \equiv (\mathfrak{M}\mathfrak{M})_*;\ J),$$

where J is a functional operation on $\mathfrak{K}$ to $\mathfrak{A}$.

This basis Σ_5 is, by the omission of the class $\mathfrak{N} \equiv \mathfrak{M}^2_*$, simpler than the basis Σ_4. Further, the equation G^4 is an instance of the equation G^5, viz., for the operation J^5 with

$$J^5_{(tu)}\varphi(tu) = J^4_t\varphi(tt)$$

for every function φ of $\mathfrak{K}$. This stipulation is legitimate since for every function $\varphi(tu)$ of $\mathfrak{K}_{tu} \equiv (\mathfrak{M}_t\mathfrak{M}_u)_*$ the reduced function $\varphi(tt)$ belongs to $\mathfrak{N}_t \equiv \mathfrak{M}^2_{t*}$. Accordingly, the general theory of the equation G^5 based on Σ_5 yields, as a special instance, a theory of the equation G^4 based on Σ_4.

Furthermore, in accordance with the derivation of the basis Σ_5, the general theory of the equation G^5 contains as an instance a theory of the equation

$$(G^{4\omega}) \qquad \xi(s) = \eta(s) - zJ^4_tJ^4_u\kappa(st)\omega(tu)\eta(u), \quad (s),$$

where ω is a function of $\mathfrak{K}$. The operation J^5 for this instance is the operation

$$J^5_{(tu)}\varphi(tu) \equiv J^4_tJ^4_u\varphi(tu)\omega(tu)$$

for every function φ of $\mathfrak{K}$. Under the postulates to be laid on the basis Σ_4 the double operation $J^4_tJ^4_u$ is applicable to the product $\varphi(tu)\omega(tu)$ of two functions of the class $\mathfrak{K}$.

It is to be noted, however, that *the geometric analogy*, let us say, *between the sphere and the ellipsoid* had as primary function not that of enabling us to treat the equation $G^{4\omega}$

based on Σ_4 as a special instance of the equation G^5 based on Σ_5; at least for the Fredholm theory, the equation $G^{4\omega}$ may be treated as the equation G^4 with the kernel $\kappa(st)$ replaced by $J^4_t\kappa(st)\omega(tu)$, since under the postulates to be imposed on the basis Σ_4 this is a function of the kernel class $\mathfrak{K}$ and the operations J^4_t, J^4_u are commutative. Its primary function was rather to lead us from the basis Σ_4 to a basis Σ_5, possessing in common with Σ_4 the closure property C_1, and possessing furthermore the *closure property* C_2—that for the basis Σ_5 a similar use of the geometric analogy leaves us on the basis Σ_5. In fact, if we seek a basis Σ for the equation G of such a nature that the corresponding inner product

$$J\alpha\beta$$

shall have as one of its instances the generalized inner product

$$J^5_{(tu)}J^5_{(vw)}\alpha(t)\,\omega(uv)\beta(w),$$

where ω is a function of the class $\mathfrak{K}$, we are led to set

$$J\alpha\beta \equiv J_{(tw)}\alpha(t)\beta(w);$$

thus the new operation J is still an operation J on $\mathfrak{M}\mathfrak{M}$ or preferably on $(\mathfrak{M}\mathfrak{M})_* \equiv \mathfrak{K}$, that is, an operation of precisely the same type as J^5.

As in the preceding remarks, we notice that the equation

$$(G^{5\omega}) \qquad \xi(s) = \eta(s) - zJ^5_{(tu)}J^5_{(vw)}\kappa(st)\omega(uv)\eta(w) \quad (s)$$

may be treated, either as the equation G^5 with the kernel $\kappa(st)$ replaced by $J^5_{(tu)}\kappa(st)\omega(uv)$, or as the equation G^5 with the operation $J^5_{(tu)}$ replaced by a new operation $J'^5_{(tu)}$ defined by the equation

$$J'^5_{(tu)}\varphi(tu) \equiv J^5_{(tw)}J^5_{(vu)}\varphi(tu)\,\omega(wv)$$

for every function φ of $\mathfrak{K}$.

8. *Resumé.*

To summarize our course to this point: We have as data the four analogous equations (I, II$_n$, III, IV) with their four analogous theories. The theories are of two stages: for brevity, F, the Fredholm theory of the general kernel, and H, the Hilbert-Schmidt theory of the real symmetric or of the more general hermitian kernel.

Guided by the heuristic principle of unification by abstraction, we formulate the general equation

$$(G) \qquad \xi = \eta - zJ\kappa\eta,$$

embracing as instances the four typical equations; and we seek the foundations, viz., the terminology or basis, and the postulates, of a general theory of the equation G which shall embrace as instances the four theories of at least the regular cases of the respective equations. This is the fundamental closure property C_1.

The form of the equation G dictates the basis Σ_1, which by consideration of the typical equations and of the obvious nature of their theories we simplify to Σ_2, Σ_3, Σ_4 in succession.

The metric-geometric analogy of the sphere and the ellipsoid leads on to the present basis

$$\Sigma_5 = (\mathfrak{A};\ \mathfrak{P};\ \mathfrak{M};\ \mathfrak{K} \equiv (\mathfrak{M}\mathfrak{M})_*;\ J \text{ on } \mathfrak{K} \text{ to } \mathfrak{A}),$$

with the general equation G interpreted as meaning

$$(G) \qquad \xi(s) = \eta(s) - zJ_{(tu)}\kappa(st)\eta(u) \quad (s).$$

And for this basis and its theory there is the closure property C_2.

The operation J is a functional operation on $\mathfrak{K}$ to $\mathfrak{A}$, that is, if φ is a function of the class $\mathfrak{K}$,

$$J\varphi \equiv J_{(tu)}\varphi(tu)$$

denotes a definite number a of the class $\mathfrak{A}$. For purposes of application J or $J_{(tu)}$ is often definable as a double operation, in the form $J_{(tu)} = J_t'J_u''$. For purposes of the general theory of the basis Σ_5, however, $J_{(tu)}$ is an *indivisible operation.* Of course $J_{(tu)}\varphi(tu)$ is equal, e. g., to $J_{(tw)}\varphi(tw)$ or $J_{(vu)}\varphi(vu)$ or $J_{(vw)}\varphi(vw)$, but it is in general not equal to $J_{(ut)}\varphi(tu)$.

9. *The Definitive Basis* Σ_6.

The basis Σ_5 is in effect definitive for the general Hilbert-Schmidt theory H. Suitable postulates will be given for both bases Σ_4, Σ_5. However, for the general Fredholm theory F the metric-geometric analogy, by leading us from the operation J_t on $\mathfrak{N}_t$ over to the operation $J_{(tu)}$ on $\mathfrak{K}_{tu}$, enables us to proceed to a still more general basis Σ_6.

In fact, for the theory H we have the respective definitions

$$\kappa(ts) = \kappa(st) \quad \text{and} \quad \kappa(ts) = \overline{\kappa(st)},$$

of the symmetry of the real-valued kernel κ and the hermitian character of the complex-valued kernel κ. These definitions require the arguments s and t to have the same range $\mathfrak{P}$.

There is, however, no such necessity in the theory F. Thus we are led to the following definitive basis for the theory F, viz.,

$$\Sigma_6 = \left(\mathfrak{A};\ \begin{matrix}\tilde{\mathfrak{P}}\\ \hat{\mathfrak{P}}\end{matrix};\ \begin{matrix}\tilde{\mathfrak{M}}\\ \hat{\mathfrak{M}}\end{matrix};\ \begin{matrix}\mathfrak{K} \equiv (\tilde{\mathfrak{M}}\hat{\mathfrak{M}})_*\\ \breve{\mathfrak{K}} \equiv (\hat{\mathfrak{M}}\tilde{\mathfrak{M}})_*\end{matrix};\ J \text{ on } \breve{\mathfrak{K}} \text{ to } \mathfrak{A}\right).$$

Here $\tilde{\mathfrak{P}}$, $\hat{\mathfrak{P}}$ are two conceptually but not necessarily actually distinct classes of elements $\tilde{p}$, $\hat{p}$ respectively; $\tilde{\mathfrak{M}}$, $\hat{\mathfrak{M}}$ are two classes of functions $\tilde{\mu}$ on $\tilde{\mathfrak{P}}$, $\hat{\mu}$ on $\hat{\mathfrak{P}}$ respectively; $\mathfrak{K}$ and $\breve{\mathfrak{K}}$ are the $*$-composites of the two classes $\tilde{\mathfrak{M}}$, $\hat{\mathfrak{M}}$ in the respective orders $\tilde{\mathfrak{M}}$, $\hat{\mathfrak{M}}$; $\hat{\mathfrak{M}}$, $\tilde{\mathfrak{M}}$—that is, the functions κ of $\mathfrak{K}$ bear their arguments $\tilde{p}$, $\hat{p}$ in the order $(\tilde{p}\hat{p})$, while the functions $\breve{\kappa}$ of $\breve{\mathfrak{K}}$ bear their arguments in the reverse order $(\hat{p}\tilde{p})$; thus there is a correspondence of the functions κ, $\breve{\kappa}$ of such a nature that for two corresponding functions κ, $\breve{\kappa}$ we have $\kappa(\tilde{p}\hat{p}) = \breve{\kappa}(\hat{p}\tilde{p})$ for every pair of arguments $\tilde{p}$ of $\tilde{\mathfrak{P}}$, $\hat{p}$ of $\hat{\mathfrak{P}}$; and functions κ, $\breve{\kappa}$ occurring together are understood to be in this sense corresponding functions, each the *transpose* of the other. Finally, J is a functional operation on $\breve{\mathfrak{K}}$ to $\mathfrak{A}$, so that for every function $\breve{\kappa}$ of $\breve{\mathfrak{K}}$ the expression

$$J\kappa \equiv J_{(tu)}\breve{\kappa}(tu),$$

denotes a definite number a of $\mathfrak{A}$. Here, in order to make the formulas based on Σ_6 readily comparable with those based on Σ_5, we agree that the elements

$$\tilde{p},\ s,\ u,\ w;$$

$$\hat{p},\ t,\ v,\ r,$$

are generic elements of the classes $\tilde{\mathfrak{P}}$; $\hat{\mathfrak{P}}$ respectively.

Based on Σ_6 we have the general pair of adjoint equations

$$(G) \qquad \tilde{\xi} = \tilde{\eta} - zJ\kappa\tilde{\eta};$$

$$(G) \qquad \hat{\xi} = \hat{\eta} - zJ\hat{\eta}\kappa,$$

with the meanings

$$(G) \qquad \tilde{\xi}(s) = \tilde{\eta}(s) - zJ_{(tu)}\kappa(st)\eta(u) \quad (s);$$

$$(\breve{G}) \qquad \hat{\xi}(t) = \hat{\eta}(t) - zJ_{(rs)}\hat{\eta}(r)\kappa(st) \quad (t).$$

Introducing the functional operation $\breve{J}$, the *transpose* of J, by the equation: $\breve{J}\kappa = J\breve{\kappa}$, for every function κ of $\mathfrak{K}$, so that $\breve{J}$ is on $\mathfrak{K}$ to $\mathfrak{A}$, we have the adjoint equation $\breve{G}^6$ in the form

$$(\breve{G}) \qquad \hat{\xi} = \hat{\eta} - z\breve{J}\breve{\kappa}\hat{\eta},$$

viz., in the form G for the transpose basis $\breve{\Sigma}_6$, which is the basis Σ_6 with the interchange of rôles of $\tilde{\mathfrak{P}}$, $\hat{\mathfrak{P}}$; $\tilde{\mathfrak{M}}$, $\hat{\mathfrak{M}}$; $\mathfrak{K}$, $\breve{\mathfrak{K}}$; J, $\breve{J}$.

The basis Σ_5 is secured from the basis Σ_6 by the supposition

$$\tilde{\mathfrak{P}} = \hat{\mathfrak{P}} \equiv \mathfrak{P}; \quad \tilde{\mathfrak{M}} = \hat{\mathfrak{M}} \equiv \mathfrak{M}.$$

Thus the theory for the basis Σ_6 has the closure property C_1. It has moreover the closure property C_2, on the understanding that the function ω or $\omega(uv)$ is any function of the class $\mathfrak{K}$; that is, the functional operation J'

$$J'_{(tw)}\varphi(tu) \equiv J^6_{(tw)}J^6_{(vu)}\varphi(tu)\omega(wv)$$

for every function φ of the class $\breve{\mathfrak{K}}$, is a functional operation of the type J^6 on $\breve{\mathfrak{K}}$ to $\mathfrak{A}$. Further, as on the basis Σ_5, the equation

$$(G^{6\omega}) \qquad \tilde{\xi}(s) = \tilde{\eta}(s) - zJ^6_{(tu)}J^6_{(vw)}\kappa(st)\omega(uv)\tilde{\eta}(w) \quad (s)$$

may be treated, either as the equation G^6 with the kernel $\kappa(st)$ replaced by $J^6_{(tu)}\kappa(st)\omega(uv)$, or as the equation G^6 with the operation J^6 replaced by the operation J' defined above.

Setting

$$\kappa\begin{pmatrix} s_1, & \cdots, & s_n \\ t_1, & \cdots, & t_n \end{pmatrix} \equiv |\kappa(s_it_j)| \quad (i,\ j = 1,\ \cdots,\ n),$$

we define the Fredholm determinant $F_\kappa(z)$ of the kernel κ and the parameter z, for the general theory of the adjoint equations G, $\breve{G}$ based on Σ_6, as follows:

$$F_\kappa(z) \equiv \sum_{k=0}^{\infty} \frac{(-z)^k}{k!} J_{(t_1s_1)\cdots(t_ks_k)}\,\kappa\begin{pmatrix} s_1, & \cdots, & s_k \\ t_1, & \cdots, & t_k \end{pmatrix}.$$

Here $J_{(t_1 s_1) \cdots (t_k s_k)}$ denotes the k-fold operation

$$J_{(t_1 s_1)} J_{(t_2 s_2)} \cdots J_{(t_k s_k)}.$$

The hth minor $(h = 1, 2, \cdots)$ has the definition

$$F_\kappa \begin{pmatrix} s_1, & \cdots, & s_h \\ t_1, & \cdots, & t_h \end{pmatrix} ; z \Big)$$

$$\equiv (-1)^h \sum_{k=0}^{\infty} \frac{(-z)^k}{k!} J_{(t_{h+1} s_{h+1}) \cdots (t_{h+k} s_{h+k})} \kappa \begin{pmatrix} s_1, & \cdots, & s_{h+k} \\ t_1, & \cdots, & t_{h+k} \end{pmatrix}.$$

The initial terms $(k = 0)$ of the determinant and of the hth minor are respectively

$$1; \quad (-1)^h \kappa \begin{pmatrix} s_1, & \cdots, & s_h \\ t_1, & \cdots, & t_h \end{pmatrix}.$$

Under the postulates to be specified below, Hadamard's theorem on determinants may be utilized to show that these series are permanently convergent power series in z, as to the hth minor for all values of the arguments $s_1, \cdots, t_h$. Further, on every finite circle in the z-plane the hth minor series converges uniformly over the composite range $\tilde{\mathfrak{P}}_1 \cdots \hat{\mathfrak{P}}_h$ relative to the class

$$\mathfrak{K}_{s_1 \cdots t_h} \equiv (\tilde{\mathfrak{M}}_{s_1} \cdots \hat{\mathfrak{M}}_{t_h})_*;$$

and its terms belong to the class $\mathfrak{K}_{s_1 \cdots t_h}$; and, as this class is closed, the hth minor for every z as a function of the arguments $s_1, \cdots, t_h$ belongs to the class $\mathfrak{K}_{s_1 \cdots t_h}$.

The general theory of the adjoint equations G, $\breve{G}$ proceeds along the usual lines. Thus, if the parameter z is not a root of the determinant $F_\kappa(z)$, the kernel κ has the reciprocal kernel λ

$$\lambda(st) \equiv \frac{F_\kappa \left(\begin{matrix} s \\ t \end{matrix} ; z \right)}{F_\kappa(z)},$$

belonging to the class $\mathfrak{K}$ and satisfying the equations

$$\kappa(st) + \lambda(st) = zJ_{(vw)} \kappa(sv) \lambda(wt) = zJ_{(vw)} \lambda(sv) \kappa(wt) \quad (st);$$

and the equations G, $\breve{G}$ have the solutions

$$\tilde{\eta} = \tilde{\xi} - zJ\lambda\tilde{\xi}, \quad \hat{\eta} = \hat{\xi} - zJ\hat{\xi}\lambda.$$

10. *Adjunctional Composition. The Closure Property C_3.*

In the algebraic domain we proceed from the single equation I

$$\text{(I)} \qquad x = y - zky$$

to the system II_n of n simultaneous equations

$$(\text{II}_n) \qquad x_i = y_i - z\sum_{j=1}^{n} k_{ij}y_j \qquad (i = 1, \cdots, n).$$

Similarly, on the basis Σ_6, we proceed from the single equation G

$$(G) \qquad \xi(s) = \tilde{\eta}(s) - zJ_{(tu)}\kappa(st)\tilde{\eta}(u) \quad (s)$$

to the system G_n of n simultaneous equations

$$(G_n) \qquad \xi_i(s) = \tilde{\eta}_i(s) - z\sum_{j=1}^{n} J_{(tu)}\kappa_{ij}(st)\,\eta_j(u) \quad (si).$$

Here we have given the parameter z, the n^2 functions κ_{ij} of the kernel class $\mathfrak{K}$, and the n functions ξ_i of the class $\tilde{\mathfrak{M}}$; and we are to determine the n functions $\tilde{\eta}_i$ of the class $\tilde{\mathfrak{M}}$. For the instance IV on the linear interval a–b Fredholm showed how to reduce the system IV_n to a single equation IV on the linear interval a–b_n $(b_n = a + n(b - a))$. A similar procedure is effective to reduce the system G_n on the basis Σ_6 to a single equation G on an enlarged basis. The procedure, however, is capable of further generalization, and, as thus generalized, of an important application to the theory of mixed linear equations.

To this end, consider a system of n bases $\Sigma_6{}^i$ $(i = 1, \cdots, n)$, viz.,

$$\Sigma_6{}^i = \left(\mathfrak{A};\ \begin{matrix}\tilde{\mathfrak{P}}^i \\ \hat{\mathfrak{P}}^i\end{matrix};\ \begin{matrix}\tilde{\mathfrak{M}}^i \\ \hat{\mathfrak{M}}^i\end{matrix};\ \begin{matrix}\mathfrak{K}^i \equiv (\tilde{\mathfrak{M}}^i\hat{\mathfrak{M}}^i) \\ \breve{\mathfrak{K}}^i \equiv (\hat{\mathfrak{M}}^i\tilde{\mathfrak{M}}^i)\end{matrix};\ J^i \text{ on } \breve{\mathfrak{K}}^i \text{ to } \mathfrak{A}\right).$$

The class $\mathfrak{A}$ is the same in the n bases Σ^i; otherwise the bases are conceptually but not necessarily actually distinct; it is however convenient to suppose that no two of the n classes $\tilde{\mathfrak{P}}^i$ have common elements and that no two of the n classes $\hat{\mathfrak{P}}^i$ have common elements; this state of affairs, being always securable by transformation, involves no restriction of generality.

As a generalization of the system G_n on the basis Σ_6 we have on the system of n bases $\Sigma_6{}^i$ the system $G_n{}^{1\cdots n}$

$$(G_n{}^{1\cdots n}) \qquad \xi^i(s^i) = \tilde{\eta}^i(s^i) - z\sum_{j=1}^{n} J^{1}_{t^j u^j}\kappa^{ij}(s^i t^j)\tilde{\eta}^j(u^j) \qquad (s^i i),$$

of n simultaneous equations. Here we have given the parameter z, the n functions ξ^i of the respective classes $\tilde{\mathfrak{M}}^i$, and the n^2 functions κ^{ij} of the respective classes $\mathfrak{K}^{ij} \equiv (\tilde{\mathfrak{M}}^i\hat{\mathfrak{M}}^j)_*$, so that $\mathfrak{K}^{ii} = \mathfrak{K}^i$; and we are to determine the n functions $\tilde{\eta}^i$ of the respective classes $\tilde{\mathfrak{M}}^i$. The n equations $(i = 1, \cdots, n)$ of the system $G_n{}^{1\cdots n}$ are to hold for every value of the respective arguments s^i of the class $\tilde{\mathfrak{P}}^i$.

We impose on each of the n bases $\Sigma_6{}^i$ the postulates to be specified below. Then the *adjunctional composite* $\Sigma_6{}^{1\cdots n}$ of the n bases $\Sigma_6{}^i$ is a basis Σ_6 satisfying the same postulates, and the system $G_n{}^{1\cdots n}$ of n simultaneous equations is equivalent to a single equation G on the composite basis Σ_6. This is the *closure property* C_3 of the theory of the linear equation G on the basis Σ_6.

For the adjunctional composite $\Sigma_6{}^{1\cdots n}$ of the n bases $\Sigma_6{}^i$ the class $\mathfrak{A}$ is the common class $\mathfrak{A}$ of the *component bases* $\Sigma_6{}^i$. The ranges $\tilde{\mathfrak{P}}$, $\hat{\mathfrak{P}}$ are the *adjunctional composites* or logical sums respectively of the ranges $\tilde{\mathfrak{P}}^i$, $\hat{\mathfrak{P}}^i$; the range $\tilde{\mathfrak{P}}^i$ is the ith *component* of the range $\tilde{\mathfrak{P}}$, and the range $\hat{\mathfrak{P}}^i$ is the ith *component* of the range $\hat{\mathfrak{P}}$. A function $\tilde{\theta}$ on the range $\tilde{\mathfrak{P}}$ determines n *component functions* $\tilde{\theta}^i$ on the respective component ranges $\tilde{\mathfrak{P}}^i$; and, conversely, any n functions $\tilde{\theta}^i$ on the respective ranges $\tilde{\mathfrak{P}}^i$ are the n components of a definite function $\tilde{\theta}$ on the composite range $\tilde{\mathfrak{P}}$; this function $\tilde{\theta}$ is the *adjunctional composite* of the n functions $\tilde{\theta}^i$. The functions $\tilde{\mu}$ on the range $\tilde{\mathfrak{P}}$ obtained thus by adjunctional composition of n functions $\tilde{\mu}^i$ of the respective classes $\tilde{\mathfrak{M}}^i$ on the ranges $\tilde{\mathfrak{P}}^i$ constitute the *adjunctional composite* $\tilde{\mathfrak{M}}$ of the n classes $\tilde{\mathfrak{M}}^i$. The classes $\tilde{\mathfrak{M}}$, $\hat{\mathfrak{M}}$ of functions $\tilde{\mu}$, $\hat{\mu}$ on the composite ranges $\tilde{\mathfrak{P}}$, $\hat{\mathfrak{P}}$ of the composite basis $\Sigma_6{}^{1\cdots n}$ are the adjunctional composites, in this sense, respectively of the classes $\tilde{\mathfrak{M}}^i$, $\hat{\mathfrak{M}}^i$ of functions $\tilde{\mu}^i$, $\hat{\mu}^i$ on the component ranges $\tilde{\mathfrak{P}}^i$, $\hat{\mathfrak{P}}^i$. The product ranges $\tilde{\mathfrak{P}}\hat{\mathfrak{P}}$, $\hat{\mathfrak{P}}\tilde{\mathfrak{P}}$ are the adjunctional composites respectively of the n^2 product ranges $\tilde{\mathfrak{P}}^i\hat{\mathfrak{P}}^j$, $\hat{\mathfrak{P}}^i\tilde{\mathfrak{P}}^j$. Then, under the postulates, the classes $\mathfrak{K} \equiv (\tilde{\mathfrak{M}}\hat{\mathfrak{M}})_*$, $\breve{\mathfrak{K}} \equiv (\hat{\mathfrak{M}}\tilde{\mathfrak{M}})_*$ of functions κ, $\breve{\kappa}$ on the ranges $\tilde{\mathfrak{P}}\hat{\mathfrak{P}}$, $\hat{\mathfrak{P}}\tilde{\mathfrak{P}}$ are the adjunctional composites respectively of the classes $\mathfrak{K}^{ij} \equiv (\tilde{\mathfrak{M}}^i\hat{\mathfrak{M}}^j)_*$, $\breve{\mathfrak{K}}^{ij} \equiv (\hat{\mathfrak{M}}^i\tilde{\mathfrak{M}}^j)_*$ of functions κ^{ij}, $\breve{\kappa}^{ij}$ on the component product ranges $\tilde{\mathfrak{P}}^i\hat{\mathfrak{P}}^j$, $\hat{\mathfrak{P}}^i\tilde{\mathfrak{P}}^j$. The functional operation J on the class $\breve{\mathfrak{K}}$ of the composite basis $\Sigma_6{}^{1\cdots n}$ is the *adjunctional composite* of the n functional operations J^i on the respective classes $\breve{\mathfrak{K}}^i$ or $\breve{\mathfrak{K}}^{ii}$ of the component bases Σ^i, viz., if the function $\breve{\kappa}$ is the adjunctional composite of the functions $\breve{\kappa}^{ij}$,

$$J\breve{\kappa} \equiv \sum_{i=1}^{n} J^i\breve{\kappa}^{ii}.$$

Now we see readily that the linear equation

$$(G) \qquad \tilde{\xi} = \tilde{\eta} - zJ\kappa\tilde{\eta}$$

for the composite basis $\Sigma_6{}^{1\cdots n}$, and the system

$$(G_n{}^{1\cdots n}) \qquad \tilde{\xi}^i = \tilde{\eta}^i - z\sum_{j=1}^{n} J^j\kappa^{ij}\tilde{\eta}^j \qquad (i = 1, \cdots, n)$$

of n simultaneous equations for the system of component bases $\Sigma_6{}^i$, are precisely equivalent. The functions $\tilde{\xi}$, $\tilde{\eta}$, κ are the adjunctional composites respectively of the functions $\tilde{\xi}^i$, $\tilde{\eta}^i$, κ^{ij}.

Accordingly, the theory of the equation G based on Σ_6 covers the theory of the system $G_n{}^{1\cdots n}$ based on the $\Sigma_6{}^i$. Thus, if the parameter z is not a root of the Fredholm determinant $F_\kappa(z)$, the kernel κ, the composite of the functions κ^{ij}, has a reciprocal kernel λ, the composite of certain functions λ^{ij}, and, corresponding to the equations

$$\kappa + \lambda = zJ\kappa\lambda = zJ\lambda\kappa,$$

we have the system of equations

$$\kappa^{ij} + \lambda^{ij} = z\sum_{k=1}^{n} J^k\kappa^{ik}\lambda^{kj} = z\sum_{k=1}^{n} J^k\lambda^{ik}\kappa^{kj} \quad (ij),$$

and for the system $G_n{}^{1\cdots n}$ we have the solution

$$\tilde{\eta}^i = \tilde{\xi}^i - z\sum_{j=1}^{n} J^j\lambda^{ij}\tilde{\xi}^j \qquad (i=1, \cdots, n).$$

For the adjoint system of equations

$$(\breve{G}_n{}^{1\cdots n}) \qquad \hat{\xi}^j = \hat{\eta}^j - z\sum_{i=1}^{n} J^i\hat{\eta}^i\kappa^{ij} \qquad (j = 1, \cdots, n)$$

we have, with the same functions λ^{ij}, the solution

$$\hat{\eta}^j = \hat{\xi}^j - z\sum_{i=1}^{n} J^i\hat{\xi}^i\lambda^{ij} \qquad (j = 1, \cdots, n).$$

The Mixed Linear Equation.—Consider a basis Σ_6 with n functional operations $J_1, \cdots, J_n$ (instead of merely one) on the class $\breve{\mathfrak{K}}$, and the corresponding mixed linear equation

$$\xi = \tilde{\eta} - z\sum_{j=1}^{n} J_j\,\kappa_j\tilde{\eta}.$$

Here the function $\tilde{\xi}$ of the class $\tilde{\mathfrak{M}}$ and n functions κ_j of the kernel class $\mathfrak{K}$ are given, and the function $\tilde{\eta}$ of the class $\tilde{\mathfrak{M}}$ is to be found.

We may treat this mixed basis as a system of n bases $\Sigma_6{}^i$, identical except that the functional operations J^i of the bases Σ^i are the respective operations J_i of the mixed basis Σ_6. Then the mixed linear equation n times repeated constitutes a particular case of the system $G_n{}^{1\cdots n}$ of n simultaneous equations on the system of bases $\Sigma_6{}^i$. Accordingly, if the parameter z is not a root of the Fredholm determinant, the system of kernel functions κ_j has, with respect to the system of functional operations J_j, a reciprocal system of kernel functions λ_j; and we have the equations

$$\kappa_j + \lambda_j = z\sum_{k=1}^{n} J_k\,\kappa_k\lambda_j = z\sum_{k=1}^{n} J_k\,\lambda_k\kappa_j \quad (j),$$

and for the mixed equation the solution

$$\tilde{\eta} = \tilde{\xi} - z\sum_{j=1}^{n} J_j\,\lambda_j\tilde{\xi}.$$

11. **-Composition. The Closure Property C_4.*

We have seen that the theory of the general linear equation G based on Σ_6 has (closure property C_1) as instances the theories of the equations (I, II_n, III, IV), and furthermore (closure property C_3) as instances the theories of the systems G_n, $G_n{}^{1\cdots n}$ of linear equations based respectively on Σ_6 and on a system of bases $\Sigma_6{}^i$. Now the step from G on Σ_6 to G_n on Σ_6 was analogous to and has as instance the step from I to II_n. The question arises whether the general theory has as instances the theories of equations arising from G by steps analogous to the steps from I to III, IV, and even to the general equation G based on Σ_6. Under the postulates to be specified, this question is to be answered in the affirmative, and this is the *closure property* C_4 of the general theory.

Consider again a system of n bases $\Sigma_6{}^i$ having the same class $\mathfrak{A}$ of all real or of all complex numbers, and otherwise conceptually, but not necessarily actually, distinct. From this system of bases Σ^i we determine the $*$-*composite basis* $\Sigma^{1\cdots n*}$ or Σ^* as follows. The class $\mathfrak{A}$ of Σ^* is the common class $\mathfrak{A}$ of the *constituent* bases Σ^i. The ranges $\tilde{\mathfrak{P}}$, $\hat{\mathfrak{P}}$ of Σ^* are the product classes respectively of the ranges $\tilde{\mathfrak{P}}^i$, $\hat{\mathfrak{P}}^i$ of the bases Σ^i. The classes $\tilde{\mathfrak{M}}$, $\hat{\mathfrak{M}}$ of functions on $\tilde{\mathfrak{P}}$, $\hat{\mathfrak{P}}$ are the $*$-composites respectively of the classes $\tilde{\mathfrak{M}}^i$, $\hat{\mathfrak{M}}^i$ on $\tilde{\mathfrak{P}}^i$, $\hat{\mathfrak{P}}^i$ of the bases Σ^i. Then the product ranges $\tilde{\mathfrak{P}}\hat{\mathfrak{P}}$, $\hat{\mathfrak{P}}\tilde{\mathfrak{P}}$ of Σ^* are the product classes respectively of the product ranges $\tilde{\mathfrak{P}}^i\hat{\mathfrak{P}}^i$, $\hat{\mathfrak{P}}^i\tilde{\mathfrak{P}}^i$ of the bases Σ^i; and the classes, $\mathfrak{K} \equiv (\tilde{\mathfrak{M}}\hat{\mathfrak{M}})_*$, $\breve{\mathfrak{K}} \equiv (\hat{\mathfrak{M}}\tilde{\mathfrak{M}})_*$ of functions κ, $\breve{\kappa}$ on $\tilde{\mathfrak{P}}\hat{\mathfrak{P}}$, $\hat{\mathfrak{P}}\tilde{\mathfrak{P}}$ are the $*$-composites respectively of the classes $\mathfrak{K}^i \equiv (\tilde{\mathfrak{M}}^i\hat{\mathfrak{M}}^i)_*$, $\breve{\mathfrak{K}}^i \equiv (\hat{\mathfrak{M}}^i\tilde{\mathfrak{M}}^i)_*$ of functions κ^i, $\breve{\kappa}^i$ on $\tilde{\mathfrak{P}}^i\hat{\mathfrak{P}}^i$, $\hat{\mathfrak{P}}^i\tilde{\mathfrak{P}}^i$ of the bases Σ^i. The functional operation J on the class $\mathfrak{K}$ has the definition

$$J\breve{\kappa} \equiv J^1 \cdots J^n\breve{\kappa}$$

for every function $\breve{\kappa}$ of the class $\breve{\mathfrak{K}}$.

This $*$-composite basis Σ^* satisfies the postulates laid on the bases Σ_6, and accordingly the general theory of the equation G based on Σ_6 has as instance the theory of the equation G for the basis Σ^*, that is, of the equation

$$\begin{aligned}\xi(s^1\cdots s^n) &= \tilde{\eta}(s^1\cdots s^n)\\ &\quad - zJ^1_{(t^1u^1)}\cdots J^n_{(t^nu^n)}\kappa(s^1\cdots t^n)\tilde{\eta}(u^1\cdots u^n) \quad (s^1\cdots s^n).\end{aligned}$$

The $*$-composition of the bases II_n and Σ_6 leads to the equation

$$\xi(is) = \tilde{\eta}(is) - z\sum_{j=1}^{n} J_{(tu)}\kappa(isjt)\tilde{\eta}(ju) \quad (is),$$

which is, notation apart, the system G_n on the basis Σ_6. In fact, the $*$-composite of II_n and Σ_6 is identical with the adjunctional composite of n systems identical with Σ_6. But the general adjunctional composition of n systems possibly distinct is not an instance of the $*$-composition here defined.

12. *Additional Definitions.*

We are to specify postulates on the bases Σ_4, Σ_5, Σ_6 enabling us to secure general theories F, H. To that end we have need of several additional definitions.

Consider a class $\mathfrak{M}$ of functions μ on the range $\mathfrak{P}$ to $\mathfrak{A}$. We have already defined the properties *linearity* (L), *closure* (C), and now define two *dominance properties* (D, D_0) and a *reality property* (R).

The function α is dominated by the function β in case for every argument p $|\alpha(p)| \leqq |\beta(p)|$. The class $\mathfrak{M}$ has the dominance property D_0 in case every function μ of $\mathfrak{M}$ is dominated by some real-valued nowhere negative function μ_0 of $\mathfrak{M}$; the function μ_0 may vary with μ. The class $\mathfrak{M}$ has the dominance property D in case for every finite or infinite sequence $\{\mu_n\}$ of functions of $\mathfrak{M}$ there is a function μ of $\mathfrak{M}$ such that the functions μ_n of the sequence $\{\mu_n\}$ are dominated respectively by certain numerical multiples $a_n\mu$ of the function μ, that is, for every n and p $|\mu_n(p)| \leqq |a_n\mu(p)|$.

A complex number a has a conjugate complex number $\bar{a}$. A function α has a conjugate function $\bar{\alpha}$, whose functional values $\bar{\alpha}(p)$ are conjugate to the corresponding functional values $\alpha(p)$, of the function α. A class $\mathfrak{M}$ of functions μ has a conjugate class $\overline{\mathfrak{M}}$ consisting of the functions $\bar{\mu}$ conjugate to the various functions μ of the class $\mathfrak{M}$. A number a is real if $a = \bar{a}$. A function α is real or real-valued if $\alpha = \bar{\alpha}$. Similarly, a class $\mathfrak{M}$ is *real* (R) if $\mathfrak{M} = \overline{\mathfrak{M}}$. Thus, the class of complex-valued continuous functions on a linear interval is real. In general, a real linear class of functions is a linear class of real-valued functions or a class of complex-valued functions whose constituent functions have real and imaginary components which range independently over a linear class of real-valued functions.

Consider a functional operation J on a class $\mathfrak{M}$ of functions μ. The operation J is *linear* (L) in case $\mu = a_1\mu_1 + a_2\mu_2$ implies $J\mu = a_1J\mu_1 + a_2J\mu_2$. The operation J has the *modular property* (M) in case there exists an *associated modulus* M, viz., a functional operation M on real-valued nowhere negative functions μ_0 of $\mathfrak{M}$ such that $M\mu_0$ is a real non-negative number, for which a relation $|\mu(p)| \leqq \mu_0(p)$ holding for every p implies the relation $|J\mu| \leqq M\mu_0$.

13. *Postulates for the Theory F.*

We secure the general Fredholm theory F of the adjoint equations G, $\breve{G}$ based on the respective systems Σ_4; Σ_5; Σ_6 by postulating that the respective classes $\mathfrak{M}$; $\mathfrak{M}$; $\tilde{\mathfrak{M}}$, $\hat{\mathfrak{M}}$ have the properties $L\ C\ D\ D_0$, and that the functional operation J on $\mathfrak{N} \equiv \mathfrak{M}^2_*$; $\mathfrak{K} \equiv (\mathfrak{M}\mathfrak{M})_*$; $\breve{\mathfrak{K}} \equiv (\hat{\mathfrak{M}}\tilde{\mathfrak{M}})_*$ has the properties $L\ M$; and for this theory we have the four closure properties $C_1\ C_2\ C_3\ C_4$; the theory F based on Σ_4 however lacks the closure property C_2.

14. *Postulates for the Theory H.*

We secure the general Hilbert-Schmidt theory H for the complex-valued hermitian kernels $\kappa(\bar{\kappa} = \breve{\kappa})$ based on the system Σ_4 or Σ_5, by postulating that the class $\mathfrak{A}$ is the class of all complex numbers, that the class $\mathfrak{M}$ has the properties $L\ C\ D\ D_0\ R$, and that the functional operation J on the class $\mathfrak{N} \equiv \mathfrak{M}^2_*$ or $\mathfrak{K} \equiv (\mathfrak{M}\mathfrak{M})_*$ has the properties $L\ M\ H\ P\ P_0$. The operation J is *hermitian* (H) in case for every two functions α, β of $\mathfrak{M}$ $\overline{J\alpha\beta} = J\overline{\beta\alpha}$, from which, in view of the properties of $\mathfrak{M}$ and the properties $L\ M$ of J, follows the relation $\overline{J\nu} = J\bar{\nu}$ or $\overline{J\kappa} = J\bar{\breve{\kappa}}$ for every function ν of $\mathfrak{N}$ or κ of $\mathfrak{K}$. The operation J is *definitely positive* ($P\ P_0$) in case for every function μ of $\mathfrak{M}$ the result $J\mu\bar{\mu}$ (for a hermitian operation J necessarily a real number) is (P) a real non-negative number (P_0) vanishing only if $\mu = 0$. Thus the operation J may be described as a *definitely positive* (PP_0) *linear* (L) *hermitian* (H) *operation having* (M) *an associated modular operation M.*

The general theory H based on the system Σ_4 or Σ_5 has the four or three closure properties $C_1\ C_3\ C_4$ or $C_1\ C_2\ C_3\ C_4$, where in C_2 the function ω of $\mathfrak{K}$ is hermitian ($\bar{\omega} = \breve{\omega}$) and positively definite, viz., for every function μ of $\mathfrak{M}$ $J_{(rs)}J_{(tu)}\mu(r)\omega(st)\bar{\mu}(u)$ is a non-negative real number vanishing only if $\mu = 0$.

If we postulate that the class $\mathfrak{A}$ is the class of all real numbers, we secure the general Hilbert-Schmidt theory H for (real) symmetric kernels $\kappa(\kappa = \breve{\kappa})$ based on the system Σ_4 or Σ_5, by imposing the same conditions as before on the class $\mathfrak{M}$ and the operation J. However there are certain simplifications. Since all the functions and the operations J are real-valued, the class $\mathfrak{M}$ is necessarily real (R); and the hermitian property (H) of J is the *symmetry* (S) $J\alpha\beta = J\beta\alpha$, holding necessarily for the theory based on Σ_4, and for the theory based on Σ_5, implying $J\kappa = J\breve{\kappa}$; and the property (PP_0) of being a definitely positive operation J is that $J\mu\mu$ is (P) a real non-negative number (P_0) vanishing only if $\mu = 0$.

We have specified the bases or terminologies and the postulates of the general theories F and H, and conclude this address on the foundations of the theory of linear integral equations with the expression of grateful appreciation of your so prolonged attention.

THE UNIVERSITY OF CHICAGO.

SHORTER NOTICES.

Lectures on Fundamental Concepts of Algebra and Geometry. By J. W. YOUNG. Prepared for publication with the coöperation of W. W. DENTON, with a Note on the Growth of Algebraic Symbolism by U. G. MITCHELL. New York, The Macmillan Company, 1911. vii + 247 pp.

THE book contains twenty-one lectures on the logical foundations of algebra and geometry in substantially the same form as delivered at the University of Illinois during the summer of 1909, with an appended note on the growth of algebraic symbolism. "The points of view developed and the results reached are not directly of use in elementary teaching. They are extremely abstract, and will be of interest only to mature minds. They should serve to clarify the teacher's ideas and thus indirectly serve to clarify the pupil's." "The results nevertheless, have a direct bearing on some of the pedagogical problems confronting the teacher." "Let the teacher be vitally, enthusiastically interested in what he is teaching, and it will be a dull pupil who does not catch the infection. It is hoped these lectures may give a new impetus to the enthusiasm of those teachers who have not as yet considered the logical foundations of mathematics." Such is the purpose of the author.

The first five lectures, of 57 pages, form an introduction which makes clear the nature of the problems to be discussed and the point of view from which they are approached. Euclid's Elements, a non-euclidian geometry, the history of the parallel postulate, the logical significance of definitions, axioms and postulates, and the consistency, independence and categoricalness of assumptions are here discussed. The example of a non-euclidean world, which is that of Poincaré in more detailed form, is a forceful means of showing the relation of our intuitional knowledge of space to an abstract geometry. The qualities of consistency, independence, and categoricalness receive emphasis.

The abstract but clear development of the cardinal, negative, rational, irrational, and complex numbers takes up several lectures. The notions of class, order, correspondence, group, variable, function, and limit are carefully developed and discussed. The lecture on limits deserves the special attention of geometry teachers because of its modern point of view. The axioms of Hilbert and Pieri are discussed in considerable detail. The discussion of spaces of four or more dimensions will be of interest in the light of recent popular papers published on the fourth dimension.

Throughout the lectures the historical development of the concepts considered is emphasized; this method of presentation illustrates the importance of the history of a subject to a teacher, for it shows that mathematics is a live and growing science and not fixed and finally determined. Professor G. A. Miller (*Science*, July 7, 1911) has called attention to two slight historical errors. Remarks on pedagogical principles found scattered through the book make it clear that the author is fully aware of the limitations to the use of purely abstract methods in elementary teaching. "No *formal* proof of any proposition should be attempted which seems obvious to the pupil without proof." "With all our insistence on the formal logical procedure, the important fact must not be lost sight of that formal logic is in only a small minority of cases the method of mathematical discovery. Imagination, geometric intuition, experimentation, analogies sometimes of the vaguest sort, and judicious guessing are instruments continually employed in mathematical research." Such statements show that the author retains a proper perspective while emphasizing the value of the abstract method. The following definition of mathematics forms the climax reached in the last lecture: "A mathematical science is any body of propositions which is capable of an abstract formulation and arrangement in such a way that every proposition after a certain one is a formal logical consequence of some or all the preceding propositions. Mathematics consists of all such mathematical sciences." The appended note treats from a historical point of view the three stages of rhetorical, syncopated, and symbolic algebraic notation.

The book is strongly commended to teachers and prospective teachers and will be very useful to those giving teachers' courses. While a few discussions, such as that of the first three known infinite cardinal numbers, may not be appreciated by those whose knowledge of mathematics is limited, yet practically no use is made of the technique of higher mathematics. The ability for abstract thinking required of the reader is considerable, but the careful reading of such a book cannot fail to have a broadening effect particularly upon those teaching elementary mathematics and in daily contact with immature minds. There is a need for scholarly contributions to mathematical literature in English sufficiently elementary in character to be useful and inspiring to progressive secondary teachers, and we believe the book under review to be just such a contribution.

ERNEST B. LYTLE.

Leçons de Cristallographie. Par G. FRIEDEL. Paris, Hermann, 1911. 8vo. vi + 310 pp. 10 fr.

Two things have been kept in mind by the author in writing this book. One is the purely utilitarian view of furnishing some knowledge of the subject to students of the mining college. The other is the purely cultural view of the subject. The latter might seem a little strange, considering the title. But from a mathematical standpoint, there is a distinct cultural value in the study of geometrical properties in crystallographic form. This becomes evident when we remember the thirty-two types of crystals and the related finite groups. We can assert from experience, having tried the experiment for several years, with freshman classes, that such a study is fully worth the time put upon it, even worth more than some other branches of elementary mathematics. After the class has drawn all the types of crystals, and manufactured one or more apiece, considering in each case the symmetry involved, the rotations possible, the way each face is produced from the original one, they are in a position to know something of group theory, at least in a very concrete form. A half-semester is ample time to cover the ground. The interest shown in mathematics from this simple study is gratifying; the improvement in geometrical imagination is valuable, and the idea of the mathematical study of essential structure is one that will go a long way in placing the correct estimate on the necessity of mathematics in a liberal education. If such a course is given by a mathematician rather than a mineralogist, and its bearing on the various problems in mathematics that it opens up is fully brought out, there are few subjects more stimulating to the student.

The text is based upon the "law of Bravais"—that crystalline structure is a finitely periodic reticulation. This law includes the law of rational indices. The first part is devoted to the study of the individual crystal, the second to the complex crystalline structures. There are two sections in the first part, the first of them being on the geometry of the crystal, the second on the physics of the crystal. In the latter the properties are classified as those due to discontinuous vectors, those due to continuous vectors. The treatment as a whole is clear and well put. The text is elementary.

JAMES BYRNIE SHAW.

Nouvelles Tables Trigonométriques Fondamentales. By H. ANDOYER. Paris, A. Hermann. 4to. xxxii + 602 pp.

IT is not easy to arouse much enthusiasm over a table of logarithms. Most of those who use aids to calculation are apt to regard them as tools necessary to the workshop, which are taken up and laid down with absolute indifference and with absolute confidence in their complete accuracy. Perhaps the latter is justified as far as the ordinary four-, five- and seven-place tables are concerned, and if this new issue were one of the many reprints that appear almost annually it would scarcely call for notice in the pages of the BULLETIN.

But there are features of M. Andoyer's work which merit special mention. The main portion of the work consists of the logarithms of the sines, cosines, tangents and cotangents to *fourteen* places at intervals of ten seconds of arc for the whole quadrant. An additional table gives the logarithms of $\sin \alpha/\alpha$, $\tan \alpha/\alpha$ for the same interval and for the first three degrees. Four further tables used in the calculations are printed: a page of logarithms to eighteen places of certain numbers; a page of formulas for the expansions of $\log \cos \frac{1}{2}\pi x - \log (1 - x^2)$ in powers of x^2 as far as x^{40}, and similar expansions for the sine and tangent; the logarithms of the trigonometric functions for every hundredth part of the quadrant to seventeen places, together with their variations as far as the latter are sensible; and finally the logarithms of the functions to fifteen places at nine-minute intervals, together with their variations per 10″ to the seventh order in the case of the cosines.

In undertaking the work, M. Andoyer had two courses open to him. One was the usual plan of getting the results from other tables by a more or less sustained effort at correction, interpolating where the older tables were not sufficiently subdivided. As he tells in his preface, there are only two original tables which aim at the degree of accuracy he wished to attain. The first is the Trigonometria Britannica of Henry Briggs published in 1633, the other the Tables du Cadastre computed between 1794 and 1799 under the direction of de Prony but never issued in printed form. Vlacq's well known tables were also published in 1633 but were only carried to ten places. All the tables of later date are founded on these. The tables of Briggs can be relied on to a unit in the thirteenth place, those of de Proney to the twelfth place.

The accuracy of observations has so much increased during the last fifty years that the common seven-place tables no longer satisfy the demands of those who have engaged in refined work, especially in astronomy. The problem is not so much that of getting the numerical value of a single function or of a few functions: in such cases one can usually adopt devices which grind out the result at the cost of trouble and time. Many of the present day problems are on a large scale. The calculations are turned over to professional computers, and special devices which a mathematician can adopt and use are frequently not well adapted for the computer who is only familiar with the ordinary methods. Extended tables and, if possible, mechanical devices are more and more sought after in order to economize time and money in scientific work, just as in business.

For various reasons, M. Andoyer decided to abandon the attempt to correct and extend the previous tables: there was no other course open but that of computing all the functions from the beginning. To anyone familiar with numerical work this seems an enormous task. One is astonished to read that the whole of it was done in less than two years without assistance! I quote from the Preface: "Les calculs nécessaires pour l'établissement des présentes tables (sauf la Table I qui n'est pas trigonométrique) ont été faits entièrement à nouveau, par moi seul, sans aucun auxiliaire, même mécanique. Un travail régulier de chaque jour, et quelque goût naturel pour les calculs numériques m'ont permis, malgré mes occupations ordinaires et malgré quelques périodes d'inactivité ou de tâtonnements, de mener à bien ma tâche sans ennui en un temps suffisamment court, de juillet 1908 à mars 1910, soit un an et huit mois." Is there a single college or university professor in this country whose ordinary duties would leave him the amount of free time necessary to carry through in twenty months a series of calculations, the results of which occupy 600 quarto pages?

Even if M. Andoyer, whose publications in the line of celestial mechanics are well known, had occupied himself solely with the tables, the fact of their completion in so brief a time would have been worthy of notice. An examination of his Introduction reveals the methods he adopted for the purpose of abbreviating the computations. A study of these, too long to explain in a review, will repay well anyone who aspires to undertake the formation of similar tables. But it is in the methods adopted to test the accuracy of the final numbers that M. Andoyer has given the best proofs. He is not content with the common method of forming differences but has made actual use of such formulas as

$$\cos\alpha\cdot\cos\left(\alpha+\frac{\pi}{n}\right)\cdots\cos\left(\alpha+\frac{n-1}{n}\pi\right)=\pm\frac{\cos n\alpha}{2^{n-1}}$$

(where $n = 4n' \pm 1$) to test the numerical values of the functions of angles separated by considerable intervals. One may without danger agree with him when he affirms that the original manuscript is absolutely free from errors. The same care has been taken with the proofs, which have been read after the final impression; the list of errata at the end contains one error—a unit in the last place of one function,—the rest being merely defects of printing and errors amongst the signs indicating the magnitude of the remainder following the fourteenth place of decimals.

If a criticism is to be made, it is in the use of the sign just mentioned. M. Andoyer, following T. N. Thiele, puts a

+ after a number when the remainder lies between a quarter and three quarters of the last unit set down. This compels anyone who wishes to use the table to insert the sign or to provide for a possible error of three quarters of a unit in his last place. The ordinary method with no sign inserted leads to a maximum error of half a unit. It seems to the reviewer to be preferable to use two signs, a plus, for example, when the remainder is between a quarter and a half, and a minus with the last figure raised one unit when the remainder is between a half and three quarters. These signs can then be dropped without adjustment and with the usual error of half a unit.

This is, however, a very small matter. The author is to be greatly congratulated on the successful completion of his task, and on producing a volume which will undoubtedly be the standard for all future tables of the logarithms of trigonometric functions. The excellent typography—an important point—should not be forgotten. The publication has been made with aid of a subvention from the Fondation Commercy by the University of Paris.

ERNEST W. BROWN.

Annuaire du Bureau des Longitudes pour l'An 1912. Paris, Gauthier-Villars.

THE present issue has its usual share of improvements and additions by the alteration to more modern data of some of the astronomical and physical constants and the inclusion of new matter. For a handy volume of reference in such matters, it is probably unsurpassed by any other of the same size and cost. But the reviewer has had cause to wonder whether some portions of the matter are really of much value. This arose from his attempt to use the table of elements of the asteroids. On account of the convenient form in which they are given, he incautiously adopted the elements for about twenty asteroids, not for statistical purposes but for individual examinations. Two of them seemed rather remarkable and the Berlin Astronomische Jahrbuch was searched for previous values. It then appeared that the periods given were erroneous; No. 318 was set down with a period of 2204 days instead of 2104 days, and No. 624 with 4229 instead of 4429 days. These are of course slips in proof-reading, but none the less disconcerting. In tables of continuous functions such errors are easily seen and corrected; one cannot do this with isolated physical constants.

The appendices contain an account by M. G. Bigourdan of the mean temperature in different parts of France. The subject is illuminated by numerous diagrams showing the mean temperatures, daily, monthly, annually, winter, etc. M. P. Hatt gives an elementary account of the method of least squares illustrated by numerical examples. One of the best features is the full index, which enables the reader to find anything desired without trouble. For the first time the phenomena are given in legal time, that is, Greenwich time, which is nine minutes, twenty-one seconds behind Paris mean time. The change is due to a law passed on March 9, 1911, a convenience for which the astronomer as well as the traveller is grateful.

ERNEST W. BROWN.

NOTES.

THE nineteenth summer meeting of the AMERICAN MATHEMATICAL SOCIETY will be held at the University of Pennsylvania on Tuesday and Wednesday, September 10–11, 1912.

THE March number (volume 13, number 3) of the *Annals of Mathematics* contains the following papers: A third generalization of the groups of the regular polyhedrons," by G. A. MILLER; "A type of homogeneous linear differential equation," by L. A. HOWLAND; "On the complete logarithmic solution of the cubic equation," by R. E. GLEASON; "The circular numbers for a plane curve," by H. T. BURGESS; "On the sum of a certain triple series," by E. W. BROWN; "A theorem in difference equations on the alternations of nodes of linearly independent solutions," by E. J. MOULTON; "Periodic quadratic transformations in the plane," by VIRGIL SNYDER; "On the reduction of a system of linear differential forms of any order," by A. DRESDEN; "On the functional equation for the sine. Additional note," by E. B. VAN VLECK.

AT the meeting of the London mathematical society held on February 8 the following papers were read: By A. C. Dixon, "Exceptions to extensions of a theorem of Jacobi's"; by W. Burnside, "Some properties of groups whose orders are powers of primes"; by G. H. Hardy and J. E. Littlewood, "Some results concerning diophantine approximations."

At the meeting of the Edinburgh Mathematical Society on March 8; the following papers were read: By G. D. C. Stokes, "An analytical study of plane rolling mechanisms"; by C. Tweedie, "The fundamental formula for the area of a triangle in analytical geometry"; by E. M. Horsburgh, "Proof of a fundamental relation between the bending moment and load curves"; by H. Levy, "An extension of Clairaut's differential equation."

The annual meeting of the British association for the advancement of science will be held at Dundee, Scotland, September 4–10, under the presidency of Professor E. A. Schäfer. Professor H. L. Callendar is chairman of section A, mathematical and physical sciences.

The thirty-ninth summer meeting of the French association for the advancement of science will be held at Toulouse, August 1–7, under the presidency of Professor Gariel. Professor E. Belot is chairman of the section of pure and applied mathematics.

In response to the wish expressed at the Milan meeting of the international commission on the teaching of mathematics for a supply depot in which all the publications of the commission can be purchased, the firm of Georg et Cie, Geneva, Switzerland, has undertaken to provide such a repository, and to keep in stock a sufficient supply to meet all regular demands. The report of the Milan meeting was published in the *Enseignement Mathématique*, November 11, 1911. It contains a list of all the publications of the commission which had appeared at that time, together with their prices. This arrangement, and the provision for receiving the reports of the American subcommission from the Bureau of Education in Washington furnish an excellent opportunity to prepare for an intelligent understanding of the full report to be laid before the fifth international congress of mathematicians in Cambridge next August.

Reports of American committees XI, "Mathematics at West Point and Annapolis," and VI, "Mathematics in the technical secondary schools in the United States" of the International commission on the teaching of mathematics have just been issued by the United States bureau of education in Washington. Copies of these and the preceding reports may be secured gratis on application.

THE twelfth regular meeting of the Rochester section of the association of teachers of mathematics in the Middle States and Maryland was held at Buffalo, March 9, 1912. The following papers were read: "Ways in which teachers of mathematics may grow," by C. N. Millard, with formal discussion by E. C. Brown and G. G. McEwen; "Notes on teaching radicals," by Miss M. M. Wardwell, with formal discussion by Miss M. E. Crofts and Miss J. C. Davis; "Suggestions on graphs in elementary algebra," by W. Betz; "Graphs in advanced secondary mathematics," by A. S. Gale. The association has other sectional organizations in New York, Philadelphia, and Syracuse, in all of which regular meetings are held.

THE firm of B. G. Teubner, in Leipzig and Berlin, has just issued a 64 page catalogue of mathematical models prepared by H. Wiener and P. Treutlein. The first part consists of 106 wire models representing quadric surfaces, helicoids, space curves, and linkages. The second part enumerates an extensive collection of wire frames and wooden models for use in solid geometry and in elementary projective geometry.

THE following university courses in mathematics are announced for the summer semester, April 15 to August 15:

UNIVERSITY OF MUNICH. — By Professor F. LINDEMANN: Integral calculus, five hours; Theory of linear differential equations and conformal representation, four hours; Quadrature of the circle, two hours; Seminar, two hours. — By Professor A. VOSS: Analytic geometry of space, four hours; Analytic mechanics, II, four hours; Seminar, two hours. — By Professor A. PRINGSHEIM: Elementary theory of ordinary differential equations, four hours; Analytic theory of numbers, four hours. — By Professor H. BRUNN: Analytic geometry and analysis situs, two hours. — By Professor K. DOEHLEMANN: Descriptive geometry, II, with exercises, six hours; Synthetic geometry, II, four hours; Imaginary elements in geometry, one hour. — By Professor G. HARTOGS: Differential calculus, five hours. — By Dr. K. BÖHM: Selected chapters of differential geometry, three hours; Theory of probabilities with application to insurance, three hours.

UNIVERSITY OF STRASSBURG.—By Professor H. WEBER: Definite integrals and introduction to the theory of functions, four hours; Algebra, two hours; Seminar, two hours.—By Professor F. SCHUR: Theoretical mechanics, four hours; Ordinary differential equations, two hours; Seminar, two hours.—By Professor M. SIMON: Stereometry of non-euclidean geometry, one hour.—By Professor J. WELLSTEIN: Elliptic and hyperelliptic functions, four hours; Theory of invariants, two hours; Seminar, two hours.—By Professor R. v. MISES: Hydromechanics, three hours; Differential and integral equations of mechanics, three hours; Seminar, two hours.—By Dr. P. EPSTEIN: Introduction to higher mathematics, two hours; Seminar, two hours.—By Dr. A. SPEISER: Theory of algebraic number fields, two hours; Recent problems in the theory of groups, two hours; Proseminar, two hours.

DR. M. RIESZ has been appointed docent in mathematics at the University of Stockholm.

DR. SLESZYNSKI has been appointed docent in mathematics at the University of Cracow.

DR. V. POLLACK, of the technical school at Vienna, has been promoted to an associate professorship of geodesy.

DR. E. T. WHITTAKER, royal astronomer of Ireland, has been appointed professor of mathematics at the University of Edinburgh, as successor of the late Professor G. Chrystal.

AT Purdue University Dr. E. G. BILL has been promoted to an assistant professorship of mathematics.

RECENT catalogues of second-hand books:—A. Hermann et Fils, 6 rue de la Sorbonne, Paris, catalogue 113, 1900 titles in mathematics, physics, and mechanics.—W. Heffer and Sons, Cambridge, England, catalogue 88, 2100 titles in mathematics and science.—Oswald Weigel, Königstrasse 1, Leipzig, catalogue 25, 600 titles in exact sciences.—Joseph Baer und Co., Hochstrasse 6, Frankfort, catalogue 599, periodicals 1700 titles.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

d'Adhémar (R.). See Montessus (R. de).

Ball (W. W. R.). Mathematical recreations and essays. 5th edition. New York, Macmillan, 1911. 12mo. 16+506 pp. $2.75

Blaubrt (M.). Ueber einige Anwendungen der elliptischen Funktionen auf die Theorie des ebenen Gelenkvierecks. (Diss.) Rostock, 1911.

Borel (E.). See Tannery (J.).

Braude (F.). Ueber einige Verallgemeinerungen des Begriffes der Mannheimschen Kurve. (Diss.) Heidelberg, 1911.

Carl (A.). Ueber höhere Rückkehr- und Wendepole. (Diss.) Jena, 1911.

Del Carral (J. I.). Nuevos metodos para resolver ecuaciones numericas. Madrid, Romo, 1911. 8vo. 303 pp.

Diophante. Le traité des nombres polygones de Diophante d'Alexandrie. Traduction française avec une introduction, par G. Massoutié. Paris, Gauthier-Villars, 1911. 8vo. 32 pp. Fr. 1.50

Düsing (K.). Die Elemente der Differential- und Integralrechnung in geometrischer Methode. Ausgabe B. Hannover, Janicke, 1911.

Eggers (G.). Ueber gewisse mit den Kegelschnitten zusammenhängende Kurven höherer Ordnung. (Diss.). Halle, 1911.

Fairons (J.). Sur les involutions du quatrième ordre. Bruxelles, Hayez, 1909. 8vo. 68 pp. Fr. 1.75

Fender (W.). Zur Theorie von verallgemeinerten Bernouillischen und Eulerschen Zahlen. (Diss.) Jena, 1911.

Frenzel (C.). Die Fundamente für eine elementare Einleitung in die Differential- und Integralrechnung. (Festschrift zur 50jährigen Jubelfeier des Gymnasiums zu Lauenburg. 2ter Teil.) Lauenburg, 1911. 8vo. 47 pp.

Galdeano (Z. de). Algunos conceptos fundamentales en un curso de análisis matemático y de las funciones. Zaragoza, Casañal, 1911. P. 2.50

Grünbaum (H.). Funktionenlehre und Elemente der Differential- und Integralrechnung. 3te umgearbeitete und vermehrte Auflage des Lehr- und Übungsbuches der Differentialrechnung. Stuttgart, Grub, 1912. 8vo. 12+196 pp. M. 4.00

Helmholtz (H. v.) Volksausgabe in einem Bande. Von L. Königsberger. Braunschweig, Vieweg, 1911. 14+356 pp. M. 4.50

Hoppe (E.). Mathematik und Astronomie im klassischen Alterthum. Heidelberg, Winter, 1911. M. 7.00

Innes (R.). A logical notation for mathematics. Kapstadt, 1911.

Jaschewski (J.). See Minkowski (H.).

Junge (G.). Ueber den Fehler bei logarithmischen Rechnungen. (Progr.) 1ter Teil. Landsberg, 1911. 19 pp.

KNESER (A.). Mathematik und Natur. (Rektorats-Rede, Breslau Universität.) Breslau, 1911.

KÖNIGSBERGER (L.). See HELMHOLTZ (H. v.).

KÖSSLER (H.). Über windschiefe Kegelschnitte. (Diss.) Halle, 1911.

LORIA (G.). Poliedri, curve e superficie secondo i metodi della geometria descrittiva. Milano, Hoepli, 1912.

MASSOUTIÉ (G.). See DIOPHANTE.

MINKOWSKI (H.). Raum und Zeit. (Russian translation by J. Jaschewski.) St. Petersburg, 1911.

MONTESSUS (R. DE) e D'ADHÉMAR (R.). Calcul numérique. 1re partie: Operations arithmétiques et algébriques. 2e partie: Intégration. (Encyclopédie scientifique.) Paris, Doin, 1911. Fr. 5.00

PLEMELJ (J.). Potentialtheoretische Untersuchungen. (Gekrönte Preisschrift.) Leipzig, Teubner, 1911. 8vo. 19+100 pp. M. 6.00

REIMERDES (O.). Die Niveau- und Fallinien auf Flächen, insbesondere auf Modulflächen analytischer Funktionen. (Diss.) Kiel, 1911.

RIEDER (K). Polynomische Entwicklungen von Funktionen einer komplexen Variablen. (Diss.) Basel, 1911.

RIEMANN (B.). See WEBER (H.).

TANNERY (J.). Science et philosophie. Avec une notice de E. Borel. Paris, Alcan, 1911. 16mo. 16+336 pp. Fr. 3.50

TIKHOMANDRITZKY (M.). Eléments de la théorie des intégrales abéliennes. Nouvelle édition, revue, corrigée, complétée de notes et en partie refaite entiérement. St. Petersburg, 1911.

WEBER (H.). Die partiellen Differentialgleichungen der mathematischen Physik. Nach Riemanns Vorlesungen. 5te Auflage. 2ter Band. Braunschweig, Vieweg, 1912. 8vo. 575 pp. M. 16.80

WILSON (E. B.). Advanced calculus. A text upon select parts of differential calculus, differential equations, integral calculus, theory of functions, with numerous exercises. Boston, Ginn, 1911. 8vo. 566 pp. Cloth. $2.75

II. ELEMENTARY MATHEMATICS.

BAKER (W. M.) and BOURNE (A. A.). A new geometry. Books 1–3. London, Bell, 1912. 8vo. 148 pp. 1s. 6d.

BENDT (F.). Grundzüge der Trigonometrie. 4te Auflage. Leipzig, Weber, 1911. 8+135 pp. Cloth. M. 2.00

BODE (P.). Die Reform des mathematischen und naturwissenschaftlichen Unterrichts an hoheren Schulen in der Gegenwart. (Diss.) Leipzig, 1911.

BORCHARDT (W. G.) and PERROTT (A. D.). Geometry for schools. Volume 3, stage 3, section 2. London, Bell, 1912. 8vo. 1s.

BOURNE (A. A.). See BAKER (W. M.).

BRUNO (G. M.). Geometria, con numerosos ejercicios. 2e edición, revisada y aumentada. Tours, Mame, 1911. 16mo. 392 pp.

DIECKMANN. Leitfaden zur Aufgabensammlung für den Unterricht in Algebra. Leipzig, Degener, 1911. M. 1.50

DONADT (A.). Arithmetik und Algebra zum Selbstunterricht. Leipzig, Brandstetter, 1911. M. 4.50

EVERETT (J. P.). See STEIMLE (C. P.).

EXERCICES de géométrie comprenant l'exposé des méthodes géométriques et 2000 questions résolues, par F. G. M. 5e édition. Paris, de Gigord, 1912. 8vo. 24+1302 pp.

GERMEAU (J.). Essai d'un cours de trigonométrie rectiligne avec de nombreuses applications. 2e édition. Namur, Wesmael-Charlier, 1911. 8vo. 98 pp. Fr. 1.50

GIRNDT (M.). Leitfaden der bautechnischen Algebra. 4te Auflage. Leipzig, Teubner, 1911. M. 1.50

—— und LIEBMANN (A.). Logarithmen und Kurventabellen für den Gebrauch an Tiefbauschulen. Leipzig, 1911. M. 1.20

GODFREY (C.) and SIDDONS (A. W.). A shorter geometry. London, Cambridge University Press, 1912. 8vo. 324 pp. 2s. 6d.

GUERBY (E.). See ROUSIER (G.).

GUICHARD (C.). Traité de géométrie. Tome 1er. 4e édition. Paris, Vuibert, 1911. 8vo. 8+566 pp.

HOCEVAR (F.). Lehr- und Übungsbuch der Geometrie für Gymnasien und Realgymnasien. 7te Auflage. Wien, Tempsky, 1912. Kr. 2.80

HOCHHEIM (A.). Aufgaben aus der analytischen Geometrie der Ebene. Heft 1: Die gerade Linie, der Punkt, der Kreis. A: Aufgaben. B: Auflösungen. 4te, vermehrte Auflage, bearbeitet von O. Jahn und F. Hochheim. Leipzig, Teubner, 1911. Cloth. M. 5.00

——. Aufgaben aus der analytischen Geometrie der Ebene. Heft 3: Die Kegelschnitte. Abteilung II. A: Aufgaben. B: Auflösungen. 2te, vermehrte Auflage, bearbeitet von F. Hochheim. Leipzig, Teubner, 1911. Cloth. M. 4.20

ISVOLSKY (N.). Treatise on geometry. I: Plane geometry. II: Solid geometry. (Russian.) Moscow, 1911. 266+127 pp.

KOMMERELL (V.) und KOMMERELL (R.). Analytische Geometrie. Für den Schulgebrauch bearbeitet. Tubingen, Laupp, 1912. 8vo. 8+180 pp. M. 2 40

LIEBMANN (A.). See GIRNDT (M.).

LIETZMANN (W.). Der mathematische Elementarunterricht und die Mathematik an den Lehrerbildungsanstalten. (5ter Band der Abhandlungen uber den mathematischen Unterricht in Deutschland.) 1tes Heft: Stoff und Methode des Rechnenunterrichts in Deutschland. Leipzig, Teubner, 1912. 8vo. 7+125 pp. M. 3.00

LORCHER (O.). Kurzgefasster methodischer Lehrgang der ebenen Geometrie nebst Grundzügen der Raumgeometrie. Stuttgart, Grub, 1912. 8vo. 8+72 pp. M. 1.20

MAINGIE (J.). Manuel d'algèbre élémentaire. Namur, Wasmael-Charlier, 1911. 16mo. 79 pp. Fr. 1.25

MATRICULATION mathematics papers. University of London. 1905–1911. London, Clive, 1912. 8vo. 112 pp. 1s. 6d.

MELDAU (H.). See SCHILLING (C.).

MILLIS (C. T.). Technical arithmetic and geometry. 2nd edition, revised. London, Methuen, 1912. 8vo. 316 pp. 3s. 6d.

MONNET (G.). See NAUD (L.).

NAUD (L.) et MONNET (G.). Douze cents problèmes à l'usage des candidats aux examens de l'administration des postes et des télégraphes. Paris, Palais des Sociétés Savantes, 1911. 8vo. 96 pp.

RODGER (J.). An introduction to the use of common logarithms. London, Blackie, 1912. 8vo. Limp. 1s.

ROUSIER (G.) et GUERBY (E.). Cours de trigonométrie, à l'usage des élèves des classes de mathématiques et de première C et D. Paris, Hatier, 1912. 12mo. 232 pp.

SALOMON (A.). Leçons de géométrie à l'usage de l'enseignement secondaire des jeunes filles. Géométrie plane. 5e édition. Paris, Vuibert, 1911. 16mo. 230 pp.

SCHILLING (C.) und MELDAU (H.). Der mathematische Unterricht an den deutschen Navigationsschulen. (4ter Band, 4tes Heft der Abhandlungen über den mathematischen Unterricht in Deutschland.) Leipzig, Teubner, 1912. 8vo. 6+82 pp. M. 2.00

SCHÜLKE (A.). Aufgabensammlung aus der reinen und angewandten Mathematik. 1ter Band. 2te Auflage. Leipzig, Teubner, 1912. M. 2.20

SCOTT (E. E.). Tables of logarithms and anti-logarithms to five places. Student's edition. London, Layton, 1912. 8vo. 390 pp. 5s.

SIDDONS (A. W.). See GODFREY (C.).

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DEFINITE INTEGRALS CONTAINING A PARAMETER.

BY PROFESSOR D. C. GILLESPIE.

(Read before the American Mathematical Society, February 25, 1911.)

A FUNCTION $f(\alpha, x)$ is defined for each pair of values of α and x in the closed region $0 \leqq \alpha \leqq 1$ and $0 \leqq x \leqq 1$. For each value of α in the interval $(0, 1)$ the function $f(\alpha, x)$ is an integrable function of x according to Riemann's definition. A function $F(\alpha)$ is thus defined by the equation

$$F(\alpha) = \int_0^1 f(\alpha, x)dx.$$

The problem considered in this paper is one of uniform convergence; namely, the determination of the conditions to be imposed on the function $f(\alpha, x)$ in order that corresponding to any positive number ϵ there exist a number δ independent of α such that

$$\text{(I)} \qquad \left| F(\alpha) - \sum_{i=0}^{i=n} f(\alpha, \xi_i)(x_i - x_{i-1}) \right| < \epsilon,$$

$$(x_0 = 0, \quad x_n = 1, \quad x_{i-1} \leqq \xi_i \leqq x_i)$$

for $(x_i - x_{i-1}) < \delta$.

Closely associated with this problem of uniform convergence are, at any rate, two others which lend interest to it. Of these, one is the problem concerning the continuity of $F(\alpha)$. Under the assumption that $f(\alpha, x)$ is a continuous function of α for each value of x, a necessary and sufficient condition that $F(\alpha)$ be a continuous function of α follows from the theory developed. The conditions under which the roots of the equation $F(\alpha) = 0$ are limiting points of the roots of the sequence of equations

$$\sum_{i=0}^{i=n} f(\alpha, \xi_i)(x_i - x_{i-1}) = 0$$

as n becomes infinite is the second problem.

The absence of continuity conditions does not preclude the existence of the inequality (I).

Example (a):

$$f(\alpha, x) = 1 + 2x \text{ for } \alpha \text{ rational},$$

$$f(\alpha, x) = 0 \text{ for } \alpha \text{ irrational},$$

$$F(\alpha) = 2 \text{ for } \alpha \text{ rational}; \ F(\alpha) = 0 \text{ for } \alpha \text{ irrational}.$$

For a fixed value of x, $f(\alpha, x)$ is discontinuous in α at every point of the interval (0, 1), and $F(\alpha)$ is also discontinuous at every point. Inequality (I) nevertheless exists.

On the other hand it is not sufficient for the existence of the inequality (I) that $f(\alpha, x)$ be limited and $F(\alpha)$ be continuous.

Example (b):

$$f(\alpha, x) = \sin\frac{2\pi x}{\alpha} \quad \text{for } \alpha \neq 0, \quad f(\alpha, x) = 0 \text{ for } a = 0,$$

$$F(\alpha) = \int_0^1 \sin\frac{2\pi x}{\alpha}\, dx = \frac{\alpha}{2\pi} - \frac{\alpha}{2\pi}\cos\frac{2\pi}{\alpha} \text{ for } \alpha \neq 0,$$

$$F(\alpha) = 0 \text{ for } \alpha = 0.$$

F(α) is therefore a continuous function of α in the interval (0, 1). Let the law of subdivision be such that the end points of the νth subdivisions coincide with those values of x for which $\sin 2\nu\pi x$ has maximum values, and let $\xi_i = x_i$. There will then exist an integer n greater than any fixed integer m, and a positive number α less than any fixed positive number δ, such that

$$\left| \sum_{i=1}^{i=n} f(\alpha, \xi_i)(x_i - x_{i-1}) - 1 \right| < \epsilon,$$

where ϵ is any preassigned positive number. Since, however,

$$\lim_{\alpha=0} F(\alpha) = 0,$$

the inequality (I) does not exist.

Nor, again, is it sufficient for the existence of the inequality that both $F(\alpha)$ be continuous in α and $f(\alpha, x)$ be a continuous function of α for each value of x.

Example (c):

$$f(\alpha, x) = \frac{\alpha^{\text{[illegible]}}}{\alpha^2 + (x - 1)^2} \text{ for } x \neq 1,$$

$$f(\alpha, x) = 0 \text{ for } x = 1.$$

$f(\alpha, x)$ is then a continuous function of α for each value of x. The function

$$F(\alpha) = \int_0^1 f(\alpha, x)dx = \alpha^{\frac{1}{2}} \operatorname{arctg}\frac{1}{\alpha} \text{ for } \alpha \neq 0,$$

$$= 0 \text{ for } \alpha = 0$$

is continuous in α throughout the interval (0, 1). Let the end points of the νth subdivision fall at

$$0,\ \frac{1}{\nu},\ \frac{2}{\nu},\ \frac{3}{\nu},\ \frac{4}{\nu},\ \cdots,\ \frac{\nu-1}{\nu},\ 1$$

and let $\xi_i = x_i$ for $i \neq \nu$ and $\xi_\nu = 1 - 1/\nu^2$; then the sum having $F(\alpha)$ for its limits is

$$\frac{1}{\nu}\cdot\frac{\alpha^{\frac{3}{2}}}{\alpha^2+\left(\frac{1}{\nu}-1\right)^2}+\frac{1}{\nu}\frac{\alpha^{\frac{3}{2}}}{\alpha^2+\left(\frac{2}{\nu}-1\right)^2}$$

$$+\cdots+\frac{1}{\nu}\cdot\frac{\alpha^{\frac{3}{2}}}{\alpha^2+\left(1-\frac{1}{\nu^2}-1\right)^2}.$$

For $\alpha = 1/\nu^2$ the last term of the sum is $\frac{1}{2}$; the sum of all the terms is greater than $\frac{1}{2}$ since all the terms are positive. The function $F(\alpha)$ is continuous and equals zero when α equals zero, the inequality (I) therefore does not exist.

THEOREM I. *If $f(\alpha, x)$ be a continuous function of the two variables α and x, the inequality* (I) *exists.*

This theorem is included in the following more general

THEOREM II. *If $f(\alpha, x)$ be continuous in α uniformly with respect to x, i. e., if corresponding to any positive number ϵ there exists a δ independent of x such that $|f(\alpha + h, x) - f(\alpha, x)| < \epsilon$ for $|h| < \delta$, the inequality* (I) *exists.*

This theorem again is a special case of theorem (III).

Theorems I and II are stated under the assumption that $f(\alpha, x)$ is an integrable function of x for each value of α. In Theorem III we drop this requirement and assume only that $f(\alpha, x)$ is a limited function of x for each α.

THEOREM III. *If* $f(\alpha, x)$ *be continuous in* α *uniformly with respect to* x, *then the upper sum* converges to the upper integral* uniformly with respect to* α.†

Since $f(\alpha, x)$ is a limited function of x for each α, the upper integral exists and a function $\bar{F}(\alpha)$ is defined by the equation

$$\bar{F}(\alpha) = \overline{\int_0^1} f(\alpha, x)dx.$$

The upper limit of $f(\alpha, x)$ for values of x in the interval (x_i, x_{i-1}) is denoted by $\bar{f}_i(\alpha)$. Unless the upper sum converges to the upper integral uniformly in α throughout the interval (0, 1), there exists a point $\alpha = b$ in this interval such that in any arbitrarily small interval about b the convergence is not uniform. It is sufficient then to establish uniform convergence in the neighborhood of b.

$$\left| \bar{F}(b + \eta) - \sum_{i=1}^{i=n} \bar{f}_i(b + \eta)(x_i - x_{i-1}) \right|$$
$$\leqq \left| \sum_{i=1}^{i=n} \bar{f}_i(b)(x_i - x_{i-1}) - \bar{f}_i(b + \eta)(x_i - x_{i-1}) \right|$$
$$+ \left| \bar{F}(b) - \sum \bar{f}_i(b)(x_i - x_{i-1}) \right| + \left| \bar{F}(b + \eta) - \bar{F}(b) \right|.$$

Corresponding to any positive number ϵ, there exists a number δ_1 independent of i such that

$$| \bar{f}_i(b) - \bar{f}_i(b + \eta) | < \frac{\epsilon}{3}$$

for $| \eta | < \delta_1$. This follows from the assumption that the function $f(\alpha, x)$ is continuous in α uniformly with respect to x. The existence of the upper integral

$$\overline{\int_0^1} f(b, x)dx$$

renders it possible to choose a number δ_2 such that

$$\left| \bar{F}(b) - \sum_{i=1}^{i=n} \bar{f}(b)(x_i - x_{i-1}) \right| < \frac{\epsilon}{3} \quad \text{for } (x_i - x_{i-1}) < \delta_2$$

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* Cp. Pierpont's Functions of a Real Variable, vol. 1, p. 337; Hobson's Functions of a Real Variable, p. 339.

† The same theorem holds, of course, for the lower sum and lower integral.

$$|\bar{F}(b+\eta)-\bar{F}(b)| = \left|\overline{\int}_0^1 f(b+\eta, x)dx - \overline{\int}_0^1 f(b, x)dx\right| \leqq \left|\overline{\int}_0^1 \{f(b+\eta, x)-f(b, x)\}\right| dx$$

$$\leqq \text{the upper limit of } |f(b+\eta, x)-f(b, x)|.$$

Hence there exists a number δ_3 such that

$$|\bar{F}(b+\eta)-\bar{F}(b)| < \frac{\epsilon}{3} \text{ for } |\eta| < \delta_3.$$

Thus we have established the existence of a number δ such that

$$\left|F(b+\eta)-\sum_{i=1}^{i=n}\bar{f}_i(b+\eta)(x_i-x_{i-1})\right| < \epsilon \text{ for } |\eta| < \delta$$

and $(x_i - x_{i-1}) < \delta$.

THEOREM IV. *If the function* $F(\alpha)$ *be continuous in the interval* $(0, 1)$, *and the function* $f(\alpha, x)$ *be a continuous function of* α *for each* x, *then under any fixed law of subdivision there will correspond to any integer* m *and any positive number* ϵ *an integer* $n > m$ *and a number* δ *such that*

$$\left|F(b+\eta)-\sum_{i=1}^{i=n} f(b+\eta, x_i)(x_i-x_{i-1})^*\right| < \epsilon \text{ for } |\eta| < \delta.$$

THEOREM V. *If* $f(\alpha, x)$ *be a continuous function of* α *for each* x, $F(\alpha)$ *will be continuous at* b *provided that, corresponding to any positive number* ϵ *and any integer* m, *there exist a number* δ *and an integer* $n > m$ *such that*

$$|F(b+\eta)-\sum_{i=1}^{i=n} f(b+\eta, x_i)(x_i-x_{i-1})^*| < \epsilon \text{ for } |\eta| < \delta.$$

The proofs of Theorems IV and V are almost identical with the proofs of the two theorems which establish the necessary and sufficient condition that the sum of an infinite series of continuous functions shall be a continuous function. These theorems in infinite series, as well as those stated here for definite integrals containing a parameter, are applications of

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[*] In the function $f(b+\eta, x_i)$ x_i may be replaced by ξ_i provided the manner of assigning ξ_i is prescribed.

the following theorem concerning functions defined by sequences of continuous functions. We assume that each of the sequence of functions $\varphi_n(x)$ is continuous in the interval (0, 1), and that $\lim_{n=\infty} \varphi_n(x) = \varphi(x)$ exists. A necessary and sufficient condition for the continuity of $\varphi(x)$ in the interval (0, 1) is that, corresponding to any positive number ϵ and any integer m, the condition $| \varphi(x) - \varphi_n(x) | < \epsilon$ is satisfied for every value of x in (0, 1), where n has one of a finite number of values all greater than m, the value to be given to n depending on the value assigned to x.

CORNELL UNIVERSITY,
March 2, 1912.

ON THE V_3^3 WITH FIVE NODES OF THE SECOND SPECIES IN S_4.

BY DR. S. LEFSCHETZ.

(Read before the American Mathematical Society, April 27, 1912.)

CUBIC varieties in four-space were first investigated by Segre, in two memoirs* which are still classic, and in which he gave a generation of those having more than six nodes, especially the one with ten nodes, while he also considered varieties containing a plane, and gave some of their properties. Castelnuovo† investigated also the V_3^3 with ten nodes, and a good account of the theory of the latter is to be found in Bertini.‡ So far as we know however, varieties having nodes for which the hypercone tangent degenerates into one cut by any V_3^1 in a cone—points which we define as nodes of the second species—have been but little considered. In a previous paper § the writer has given the maximum of these nodes for surfaces, or rather a method for obtaining it. This method admitted of an evident extension to n-space, and in particular gives for V_3^3 in four-space, a maximum of these nodes equal to half the number of absolute invariants of the most general V_3^3, that is five. We propose, in the present paper, to show briefly the existence of this variety having five such nodes.

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* "Sulle varietà cubiche," *Memorie dell' Academia di Torino*, ser. 2, vol. 39 (1888). "Sulla varietà cubicha con 10 punti doppi," *Atti di Torino*, vol. 22 (1887).

† "Sulle congruenze dell 3° ordine," *Atti dell' Ist. Veneto*, ser 6, vol. 6 (1888).

‡ Geometria proiettiva degli iperspazi, p. 176.

§ "On the existence of loci with given singularities." Read before the Poughkeepsie meeting of the Society, Sept. 12, 1911.

The equation of a V_3^3 having 5 nodes, one at each of the vertices of the pentahedron of reference, is

$$\sum_{i,k=1}^{i,k=5} \frac{a_{ik}}{x_i x_k} = 0 \qquad (a_{ik} = a_{ki}, \quad a_{ii} = 0).$$

The hypercone tangent to the V_3^3 at the point (0, 0, 0, 0, 1) has for its equation

$$\sum_{i,k=1}^{i,k=4} \frac{a_{ik}}{x_i x_k} = 0,$$

and its section by a V_3^1 will be a cone if we have

$$\begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{12} & 0 & a_{23} & a_{24} \\ a_{13} & a_{23} & 0 & a_{34} \\ a_{14} & a_{24} & a_{34} & 0 \end{vmatrix} = 0.$$

If we apply now the transformation

$$X_1 x_1 = X_2 x_2 = X_3 x_3 = X_4 x_4 = X_5 x_5,$$

our V_3^3 becomes

$$\sum_{i,k=1}^{i,k=5} a_{ik} X_i X_k = 0,$$

that is, a V_3^2, which will pass through the five vertices of the pentahedron of reference, and will be cut by $X_5 = 0$ in a quadricone on account of the vanishing of the above determinant. It will therefore be tangent to $X_5 = 0$. It follows that to show the existence of a V_3^3 with five nodes of the second species, we need only construct a V_3^2 both inscribed and circumscribed to a pentahedron formed by five V_3^1. For this purpose let us consider an arbitrary V_3^2, and three arbitrary V_3^1 tangent to it, which we will denote by A, B, C. We want to find two more V_3^1, say D and E, such that together with the three given ones they satisfy the required condition with respect to V_3^2. Let L be the right line intersection of A, B, C; P_a, P_b, P_c the planes (BC), (CA), (AB); and C_a, C_b, C_c the conic intersections of these planes with V_3^2. These three conics have in common the points α, β where L cuts V_3^2. With the conditions imposed it is clear that D must go through α (or β) and E through β (or α). . . . Suppose then that D goes through α and E through β. D and E are to be determined by the conditions that they shall be tangent to V_3^2 and such that the plane (DE) shall meet C_a, C_b, C_c respectively in one point. Consider now a fixed point a on C_a, and a point b on C_b. There are two V_3^1 going through a, b, α and tangent to V_3^2, and if c is the point other than α where one of them meets C_c, there are two V_3^1 tangent to V_3^2 and going through a, c, β. If b' is the point where one of them cuts C_b, it is seen at once that (b, b') are in (4, 4) correspondence, and for any of the 8 coincidences it is evident that we have two hyperplanes D, E which together with (A, B, C) form a system of the kind required.

It may be remarked in passing that

$$\prod_{i=1}^{i=4} x_i \sum_{k=1}^{k=4} \frac{a_k}{x_k} + x_5^3 = 0$$

represents a V_3^3 with four nodes of the second species, and is a mere generalization of the cubic surface with three such nodes represented by

$$x_1x_2x_3 + x_4^3 = 0.$$

We reserve for a later occasion the consideration of the special cases that may arise in the construction given above.

LINCOLN, NEB.,
January 22, 1912.

WHAT IS MATHEMATICS?

Principia Mathematica. By ALFRED NORTH WHITEHEAD and BERTRAND RUSSELL. Vol. I. Cambridge, 1910. Royal 8vo. xvi + 666 pp. $8.00.

THE game of chess has always fascinated mathematicians, and there is reason to suppose that the possession of great powers of playing that game is in many features very much like the possession of great mathematical ability. There are the different pieces to learn, the pawns, the knights, the bishops, the castles, and the queen and king. The board possesses certain possible combinations of squares, as in rows, diagonals, etc. The pieces are subject to certain rules by which their motions are governed, and there are other rules governing the players. A treatise on chess contains all these. Further however it also contains openings which have been found to be advantageous to one or the other of the players and usually contains also various endings of games for the tyro to analyse, in order that he may see how to acquire skill in foreseeing the situations that may arise in any game. One has only to increase the number of pieces, to enlarge the field of the board, and to produce new rules which are to govern either the pieces or the player, to have a pretty good idea of what mathematics consists.

In mathematics the game is much more complicated. The pieces we handle are the members of ranges of a more or less elaborate character. These members may be numbers, functions, lines, operations, any set of things we can define.* The moves on the board are groups of operations that may be performed upon these ranges and their members. We also must take into account a feature which is present in the game of chess in one move only—that of castling. In mathematics we may handle whole combinations of elements and operations upon them, as if they were single things. That is, we must take into account complexes of operations and ranges.

With these elements we do different things, according to our taste and ability. First of all there are the developments of *structure*. These include the construction of magic squares, and other questions of tactic, arrangements and combinatory analysis, factoring, decomposition of fractions, congruences, residues, and theory of form, through the structure of groups, up to finite fields, multiple algebra, calculus of operations, symbolic logic and general algebra. Then there are developments of the *invariants* that occur in different structures. We study algebraic and arithmetic forms, group characters, projective geometry, differential forms, topology, geometry in general, and operational invariants. (We mention necessarily only a few sample cases of the problems referred to.) There is also the study of correspondences of various types, the whole field of analysis or study of *functions*. In this are such things as the functions of a real variable, trigonometric series, algebraic functions, general analysis, geometrical transformations, automorphic functions, calculus of variations, functions of a complex variable, vector fields, differential geometry, and rational mechanics. Further, and most difficult, are the studies in *inversion*. It is here that the expansions of the mathematical game take place. In this line of investigation we find algebraic corpora, ideals, modular systems, differential equations, integral equations, functional equations, and inverses of all kinds. Any theory of mathematics that would be complete is forced to account for all these different studies. To consider, for example, that one has laid the foundations of mathematics when he has produced the irrational number, is to confuse the theory of real variables with mathematics. Important as the problems of the continuum may be, the continuum is not the basis or foundation of structure. A knowledge of the different grades of ensembles does not enable one to ascertain whether a group is compound or not, or whether a number is prime or not. It does not determine the list of invariants of the decic, nor does it develop the differential parameters of a differential form of the fifth degree. The continuum has little to do with the properties of automorphic functions as functions. Cardinal and ordinal ranges do not play a prominent part in modular systems, nor in algebraic ideals, nor in functional equations. Neither likewise does a set of postulates for geometry, or some type of geometry, assist in determining how many associative algebras there are, built on twenty-four units. Indeed the postulates for associative algebras in general do not do this. So it becomes evident that when one wishes to discuss the principles of mathematics he must state what it is he refers to. If the analysis above (which was only indicated in a broad way) is correct, he may discuss the *ranges* with which mathematics has to deal, or he may discuss *operations* in general, or the principles may be those at the base of *multiple algebra*. He may mean the principles of mathematical composition and *form*, or the principles of the *invariance* in the transformations of forms, or the principles of *functionality* and correspondence in general, or the principles upon which may be founded the theory of *inversions*. For example, Russell's Principles of Mathematics was unable to handle the problem of the introduction of the imaginary into mathematics, and endeavored to crowd the whole theory of hypercomplex numbers into the theory of dimensionality. Peano's Formulaire did better, for the imaginary is defined at least by one of its examples. But the imaginary and the quaternion, and all other associative hypercomplex numbers, not to speak of those not associative, receive scant recognition in either case. A Principia Mathematica should cover the field, or it ceases to justify its title.*

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* Cf. E. H. Moore, "Introduction to a form of General Analysis," New Haven Math. Colloquium, 1910, page 2.

Further we must not confuse mathematics and mathematical reasoning. It is true we infer in mathematics. But we also infer in physics, and history, and in daily life. Mathematics has no copyright on the process. To define mathematics as the science that draws necessary conclusions,† or as the class of all formal implications,‡ does not define at all. Other branches of human learning draw necessary conclusions, and formal implication is not unknown to them. Mathematics also uses the constructive imagination, the generalizing power, the intuition, and other mental processes. To define mathematics as "the study of ideal constructions (often applicable to real objects) and the discovery thereby of relations between the parts of these constructions, before unknown," is better.§ This definition brings out the *ideal*, the *construction*, and the *discovery*, three features which are essential to mathematical development. For example, an abstract group is ideal. Functions of the roots of an equation are ideal constructions. The discovery lies in seeing that the properties of the one will explain the relations involved in the other. Again, algebraic numbers are ideal. We construct with them domains of rationality, and arrive at the Galois theory in the relations.

However attention should be drawn to the fact that among the ideal elements with which we deal are many we invent or create entirely new. Examples are easily found. Quaternions were the result of Hamilton's attempt to extend the number field. The non-euclidean geometries were the attempt to create a new geometry. If any one fact stands out prominently in mathematical investigation it is this fact of the creation of new realms of investigation. Whether these ever are applied to real objects is a matter of less importance mathematically. It is a simple affair to invent even a new logic and mode of inference. Thus, let us imagine that the contradictive process were of period three, in place of two. That is, to the proposition p there is a first contradictory p', whose first contradictory is p'', the contradictory of the last being again p. What becomes of the conventional logic now? Yet by symbolism we can develop this kind of logic as well as any. This process we may call the mathematicising of logic. Indeed the volume before us really does something of this kind for logic, as the doctrine of types is close to the example above, and the result is labeled mathematical logic.

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* Cf. Poincaré, *Rev. Mét. et Morale*, vol. 13 (1905), pp. 815–835; vol. 14 (1906), pp. 17–34, 294–317, 866–868. Hilbert, Verh. III. Int. Math. Kongr. Heidelberg (1904), pp. 174–185. Russell, *Rev. Mét. et Morale*, vol. 14 (1906), pp. 627–650. Bôcher, BULLETIN, vol. 11 (1904), pp. 115–135.

† B. Peirce, *Amer. Jour. Math.*, vol. 4 (1881), p. 97.

‡ B. Russell, Principles of Mathematics, p. 3.

§ Century Dictionary (C. S. Peirce).

The outcome is interesting to mathematicians for several reasons. First of all it is a very general or abstract branch of mathematics. Secondly this book whose first volume is under consideration takes the place of a second volume of the Principles, in which the attempt was made to reduce all mathematics to symbolic logic, or as it is now called logistic. It is expected to demonstrate formally from these notions the derivation of the properties of cardinal and ordinal integers, irrationals, series, and eventually geometry and dynamics. We desire to examine the book from a purely mathematical point of view as to the success of the attempt.

There is an Introduction of three chapters. In some cases the fuller development farther on must be read in order to see exactly what the explanations of the Introduction mean. The first chapter gives a preliminary explanation of the notations used. These symbols are able to be themselves the elements of the entire development, and are thus very fundamental. A complete table is given here for the convenience of readers.

$\vdash$	it is true that..
*	chapter
$\sim$	contradictory of..
v	..or..
. $\therefore$	and..
$\supset$	..implies that..
$\supset_x$.	implies that for x, formal implication
= .	is identical with..
= . *Df*	nominal definition
$\equiv$ ..	is equivalent to..
$\equiv_x$.	is equivalent to.. for x, formal equivalence
P_Δ	P-selections
$1 \rightarrow 1$	one-one correspondence
$\dot{-}$	negative of relation ()
$\subset$·	relation () contained in..
$\vert$	relative product of.
R'	the referent as to relation () of
$R''\ \alpha$	the referents of class α
$\breve{R}$	converse relation of R
$\exists!$	there is a case of relation
Λ	null-relation
$\dot{V}$	universal relation
⌉	relation with domain limited to a class
⌈	relation with converse domain limited to a class
$\triangleright$	relation with field limited to a class

$1 \rightarrow \mathrm{Cls}$	one-many correspondence
Cls	name for classes
Rel	name for relations
Cnv	converse
Ex	existent
Cls^2	class of classes
D‘	domain of relation..
ᗡ	converse domain of .
C‘	field of..
$\rightarrow$, *sg*,	referents of..
$\leftarrow$, *gs*,	relata of .
x ♀	operator of x on ()
♀ y	operator of () into y
α ♀ y	values of x ♀ y for x over range α
Rl	subrelation
Rl ex	existent subrelation
φa	proposition about a
φx	proposition about a variable
$\varphi \hat{x}$	propositional function
$\varphi ! \hat{x}$	predicative function
$(x) \cdot \varphi x$	proposition is true for all individuals x
$(\exists x) \cdot \varphi x$	proposition is true for some individuals x
$(\text{℩} x) \varphi x$	the x with property φ
$\hat{x}(\varphi x)$	class defined by φ
Type	range of x such that φx is significant.
$\dot{\cap}$	relation () and relation ()
$\dot{\cup}$	relation () or relation ()
$\downarrow$	couplet relation, vid.
$\cap$	common subclass
$\cup$	common superclass
$-$	negative of a class
$\subset$	contained in .
V	universal class
Λ	null class
$\exists !$	there is a member of
E!	the member of .. exists
1	the class of unit classes
$1(\alpha)$	the class of unit classes of type α
$\varphi, \psi, \chi, \theta$,	functional signs
℩	the ..
ϵ	is a member of the class determined by .
ι	class of one member, unit class.
$\iota‘\alpha$	the only member of α
Greek capitals,	constants
Small Greek letters,	usually classes
Capital italics,	variable relations
p, q, r	propositions
f, g	functions
$t‘x$	type in which x is contained
$t_0‘\alpha$	type in which α is contained
$\mathrm{p}‘\kappa$	product of classes
$\mathrm{s}‘\kappa$	sum of classes
Cl	subclass
Cl ex	existent subclass
‘	of

These symbols may be viewed in two different ways. They may be looked upon as furnishing a system of short-hand, or pasigraphy and stenography combined, intelligible to the initiated, and not only abbreviating the writing, but furnishing a mode of expression in which the usual color, shading of meaning, and associations of words are missing. This in itself would justify their use. Or we may look upon them as being symbols for the abstract elements of reasoning which have been found by the analysis, and for which no appropriate name exists. The latter formal view would not be taken by those who dislike to think that mathematics is the theory of certain combinations of symbols. But it seems to us that if we take the formal point of view, we are doing no more than when we define a rational fraction as a couple of numbers, subject to certain rules, and are able then to identify integers with those couples whose second number is 1. The foundations of arithmetic become solid. So too here if we consider that these symbols themselves, as representative of certain well-defined terms, are under consideration, we shall find a gain in clearness. In fact obscurity arises easily if we try to interpret some of the statements of the book in other ways.

We will undertake to give some notion of the ground covered in this first volume. All we can do of course in our limits of space is to discuss some of the prominent features of the book. The first thing we must consider is the meaning attached to certain words that are not used in the usual sense. The one we encounter at the beginning is the term *implication.* To this we need to devote a careful study, for it is the real basis of the further development.

Implication.

The startling statement is made early that "Newton was a man" and "The sun is hot" are equivalent propositions. From the definition of implication we see also that "Newton was not a man" implies that "The sun is hot," and "The sun is cold" implies "Newton was a man," or $2 + 2 = 4$, or "John Smith killed Pocahontas." This is a very different thing from what most of us would naturally call equivalence or implication. Implication, as used here, is a relation between two elementary propositions, or statements about particular individuals. We might raise the question as to whether there are any such propositions after all. But accepting for the time the assertion that they exist, implication is merely the statement that either the first proposition is false or else the second one is true. Now in the highly special sense in which we find terms used throughout the book, we feel instinctively that there is a certain artificial quality about every definition given or term used. Thus while we find the terms *true, false,* and *not true,* used, as well as *truth-value,* we find no real explanation given of the meaning of these words. Indeed we find later that there is a varying truth dependent upon the order of the statement. We must conclude then that the text throughout is concerned with a certain quality of the propositions considered, and not with the propositions themselves. When the authors talk about p and q they do not mean to discuss the significance of p and q but only a certain quality of p or of q. If the argument is about "Newton was a man," any other true proposition would do as well, for example, $2 + 2 = 4$. The assertions are not about the content of the propositions in either case but about the quality attached to either called its "truth-value." All propositions with the same truth-value are equivalent. They may in any implication be substituted, one for another. This explains how it is that "Newton was a man" and "The sun is hot" are equivalent. It also shows that the first implies the second, for either the first, irrespective of its significance, has the truth-value falsehood, or else the second has the truth-value truth. It is explained later (page 120) that we may throughout substitute the number 1 for any proposition with the truth-value truth, and the number 0 for any proposition with the truth-value falsehood, and reduce all formulas to the arithmetic of 0 and 1. The notion of truth-value is due to Frege,* although something similar is to be found in Boole.†

With regard to this view, we might suggest that whether the symbols p and q are to be regarded as equivalent or not depends upon the relationship they possess to other things in the universe, as well as upon their own significance. It would seem better to have used a different term to designate what the authors have in mind. A specific symbolism will make the matter clear. Let us agree to mark every proposition either with ° or with ′. This property of being tagged we will call T, thus

T (Newton was a man) is ′, T (The sun is cold) is °.

We may then state that what is meant by implication is one of the alternatives

$$\mathrm{T}(p) \text{ is }^\circ \text{ and } \mathrm{T}(q) \text{ is }^\circ, \text{ or } \mathrm{T}(p) \text{ is }^\circ \text{ and } \mathrm{T}(q) \text{ is }', \quad \text{or } \mathrm{T}(p) \text{ is }'' \text{ and } \mathrm{T}(q) \text{ is }''.$$

It is to be observed that T(q) is tagged ′ if T(p) is ′, otherwise it may be either. The one case excluded is evidently: tag of p is ′ and tag of q is °. We may admit that this is a simple thing, but (while it may be an idiosyncrasy on our part) it does not seem to be elementary. The notion of two tags, and of the property of being tagged, are clearly involved, and unless we make the tagging a purely haphazard affair, there is also involved the problem of determining *which tag must be placed on a given symbol.* The basal assumption (not mentioned) of the entire book seems to be, that we are in a position to say with regard to any proposition p about some specific thing whether it is true or not. Often it is the truth that we are endeavoring to discover. In fact, we have here the first example of the assertion that some parts of the book become more clear if we treat the results as purely symbolic, the whole being a calculus of symbols. If we know how to tag q there is no use in mentioning p. If p happens to be false, or if p happens to be true, we still confront the fact that we must decide that we have to choose between exactly the alternatives stated above. Hence for practical inference, this kind of implication seems to us to be worthless, and we therefore think another name would be desirable. The definition of implication, or rather the meaning given the implication sign, by Peano,* seems to be more fundamental, and elementary. It reads thus:

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* Russell, "The theory of implication," *Amer. Jour. Math.*, vol. 28 (1906), p. 160.

† Laws of Thought, 1854, p. 70.

$$p)q$$

de p on déduit q; si p, alors q; la p a pour conséquence la q; la q est une conséquence de la p; la p est une condition suffisante de la q; la q est une condition nécessaire de la p.
We deal here directly with the propositions p and q and not with any functions of qualities they may possess. From the formal point of view, however, we have obtained a two-valued function of the indefinite set of marks, p, q, r, etc., that is, T( ) is $'$ or $^\circ$.

The propositional form that has a variable argument is generally expressed by φx, and we come next to the implications corresponding, that is, to *formal implication.* According to the Principles of Mathematics, the formal implication is the main thing in mathematics. It means that in φx we substitute for x any symbol (later it is restricted to a given type). The propositions resulting will each have a tag, as $'$ or $^\circ$, that is, as true or false.† The same is done with ψx, where ψ is a form into which we put the variable x in order to arrive at a proposition. Each of these propositions is tagged. Then φx implies (formally) ψx if in each case the tag on φx is $^\circ$ or else the tag on ψx is $'$, the same x occurring in each of the two. In other words, we must be able to assign for any given x the proper tag for φx and for ψx, and if we make out a three column table in which the first column is marked x, the second φx, the third ψx, and then enter the values of x as arguments, and the tags as values in the proper columns, we will necessarily have in the second column both ° and ′, and in the third column also both ° and ′, but with ′ in the second will always come ′ in the third. We might also state it thus: ψx must be true at least whenever φx is true. It is explained that we do not need to know or to produce every x about which the proposition may be stated. The φ and ψ are taken intensively. This is the first case we note in which there seems to be a lack of agreement between the theory and the practice. The definition calls for a comparison of the tags on two sets of expressions, which from their character would usually be infinite in number. As the direct comparison is impossible, the practical application goes back to a problem in intension, a term the authors endeavor to the utmost to shut out of the book. Again, if φx happens to be false for x in every case, we nevertheless have ψx implied. This may be true, that from false propositions anything may be concluded, but it does not advance mathematics very much. It would seem therefore that the attempt to found the whole system on the principle of truth-values (which we have called tags) is not so very successful, and that it would be better to make the undefined implication the base of the system. Indeed the authors apparently fall into this habit unconsciously. Thus we find as one of the assertions of the book

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* Formulaire, 4 éd., page 4; 5 éd., p. 3.

† If absurd, they are not propositions.

$$*2.04 \vdash :.p.).q)r:):q.).p)r,$$

which they interpret: if r follows from q provided p is true, then r follows from p provided q is true. This reversal of conditions in the theory of functions would work havoc only too frequently. Of course the reading should be: consider {the tag on p is ° or else the tag on the statement [either the tag on q is ° or that on r is ′] is ′}, then if we mark all that has just been stated in { } with ′, we must mark also with ′ all that follows, viz. {the tag on q is ° or else the tag on the statement [either the tag on p is ° or else the tag on r is ′] is ′}. This is quite different from the Peano reading given just above.

We have dwelt upon the idea of implication as set forth here because this idea seems to be used more as a test of the accuracy of the results obtained than as a working notion. It is held in reserve as a court of last appeal. If one starts in directly with Section A and not with the Introduction, he does not encounter the notion of truth-value until *4.01 on page 120. If implication is taken as the fundamental notion and left undefined, we can define all the other symbols in terms of it and contradiction. This was done in the Principles, although the method used seems unnecessarily cumbrous. However, truth-value does not appear in the symbolism, and we have practically gained the following fundamental symbols:

φa, φb, etc., definite propositions about constant subjects a, b, etc.

φx, ψx, etc., definite propositional functions but variable arguments.

$)\cdots$, a relation between propositions, called implication.

Propositional Functions.

The propositional function is very important. It not only includes the usual predicate but may be any kind of a form with a blank place left for the entry of the argument. It is a symbol for the process that enables one to pass from a given argument term to another, the value term. The notation is as follows:

φa, a has the property φ, or of a we may say φ. This sentence is the value of φ for a.

φx, the propositional function applied to a variable argument. This is a symbol for any *one* of the values of the function, including statements which are not true as well as true statements.

$\varphi \hat{x}$, the function itself, as function. The $\hat{x}$ appears merely to assure the reader that the function really is a function of something or other. If the authors could have brought themselves to accept the Frege* notion and symbol $\varphi()$, the apparent argument could have been omitted.

$(x)\cdot\varphi x$, the entire list of values of φx are represented by this sign. In a large majority of them the truth-value would be ° of course. Also the x is restricted to the range called the type of φ. The expression reads "φx is every case where x belongs to the type of φ."

$(\exists x)\cdot\varphi x$, there are values of x which give φx the truth-value ′.

$\hat{x}(\varphi x)$, this symbol seems to have two meanings, at war with each other. In the early part of the book it is defined to mean the class (aggregate, ensemble) which consists of those arguments that make φx true. In other words, to be the set of individuals in the class determined by, or defined by, the function φ. Later it is identified with a symbol $\psi ! \hat{x}$ which is purely a function symbol, and does not represent individuals at all. This symbol practically defines the class property. The latter meaning seems to be the one which the authors expect to use, and may be interpreted to be the class as class, and not as individuals, but considered as a *denoting* symbol. This use of the symbol to represent the predicative function that would define the class collectively seems to be necessary in the system to enable us to use classes as arguments of functions. It is explained that we do not arrive at real classes thus, but only incomplete symbols. For example it is something like this. If we desire to say "The governors of states all wore silk hats," we must recast the statement to read "Certain persons were silk-hatted governors of states." This use of the class symbol $\hat{x}(\varphi x)$ and the function symbol $\varphi \hat{x}$ is close to that of the phrase "governor of a state," in two different senses, one meaning defining the qualifications necessary to be the governor of each of the states, the other defining the actual governors, so that they could be identified among other men. Both are functions. On this basis there are *no classes*, although the word class appears everywhere in the book. However, the claim is made that we have something just as good as a class, and in fact (page 84) "in mathematical reasoning, we can dismiss the whole apparatus of functions and think only of classes as "quasi-things," capable of immediate representation by a single name." An example would be the imaginary points of a curve.

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* See Principles, page 505.

However that may be as a matter of interpretation, we at least have arrived at two more symbols, from a mathematical point of view: the proposition as function, and the definition of solutions of a proposition. We have, in brief, isolated the function sign φ, and we can speak of "x such that φx." It would seem now that the fundamental thing after one has exhibited his set of elements with which he proposes to work, would be to consider functions of one variable, then synthetic processes by which these may be built up into useful structures. The importance of the propositional function is sufficiently insisted upon, but the uselessness of a mere set of isolated individuals is not dwelt upon. Nor is the character of synthetic processes in general examined at all. Of this we shall have something further to say.

Extensions.

Considerable stress is laid on the assertion that mathematics is primarily concerned with extensions, despite the fact that classes have become mere ghosts of themselves. An example would be any geometric theorem from Euclid. As "Let triangle ABC be isosceles, then triangle ABC has equal base angles." We are apparently discussing the single triangle ABC, but in reality we expect what we say to hold good for any triangle and thus for every triangle. We have the formal implication, which holds between the two sets of elementary propositions, one for each and every triangle. The ground of the reasoning seems to be an ambiguous case, and we seem to reason from any one to all. But is it so? If in a complex mental structure one chooses to pay attention only to certain features of the structure, and discuss them, ignoring the other accompanying features, are the statements about the whole complex structure, or about the portion abstracted? In proving the theorem cited above, does the color of the crayon used in drawing the triangle also enter the argument? Does the size of each individual angle and side also enter, or only certain relations they have? In finding the limit of the expression $2-(\frac{1}{2})^n$ do the particular values that n may take enter the argument at all? If one must answer no in these cases, that is to say, if one can abstract at all and reason about his abstraction, then we see no force in the constant appeal to extension. Is not the propositional function of the nature of an invariant rather? We say: *this* triangle has two equal sides, so *that* triangle, so also *yonder* triangle. In all the propositions of this sort that we choose to write down, we find the invariant phrase: two equal sides. It does not appear to be essentially different from any other invariant. If we were to conceive a transformation that could convert *this* into *that*, and *this* into *yonder*, the invariant of the transformation would in this case be: two equal sides, that is, isoscelism. Now the real question is, whether we can discuss invariants apart from the other circumstances in the concrete cases in which they are invariant. We certainly do this in mathematics. Indeed, is not this present book an attempt to discover what are the logical invariants in mathematics? In mathematics we are dealing with abstract elements all the time. The world of mathematics is an ideal, that is, an abstract world. We either abstract it from what we find in experience, or we create it de novo. For example, no such thing as *two* exists outside a mathematical mind, any more than the rainbow exists as color out there in the sky. We build our own structures and determine their relations, and no abstraction can be said to be more abstract than any other. In Socrates is mortal, Aristotle is mortal, Charlemagne is mortal, Socrates, Aristotle, and Charlemagne are abstract. No one would call them real men. They are as abstract as mortal, and as abstract as all men. Socrates is a word which represents the persistence of certain qualities of something day after day, and is thus in itself an invariant and therefore an abstraction. In one sense the whole of mathematics is the study of invariants.

If this analysis is correct, then when we say that the property φ implies the property ψ we do not need an extension to which to refer it. It is immaterial whether the extension is there or not. We can study the property as well in one case, as in a million, or an infinity of cases, if only we can isolate it itself. That is what we do in geometry, and in fact all through mathematical thinking. From this point of view (which may coincide with that which the authors denominate the philosophical, and with regard to which they admit the contention) mathematics is more concerned with intensions than extensions. These remarks apply to the extensional functions of functions. Indeed the paragraph in the middle of page 77 says:

". . . the functions of functions with which mathematics is specially concerned are extensional and . . . intensional functions of functions only occur where non-mathematical ideas are introduced, such as what somebody believes or affirms, or the emotions aroused by some fact."

It is difficult to see in this any more than the assertion that mathematics is not concerned with non-mathematical ideas, which no one would pretend to deny, unless it be some philosophers who thought that the Principles of Mathematics did not discuss mathematical ideas.

But there is a further point that we must notice. It is that the definition of function and class does not really produce the members of the extension at all. If we speak of the points of intersection of $x^2 + y^2 = 25$ and $x^2 + y^2 = 36$ what points are given as the extension of this proposition? If we speak of the roots of the equation

$$x^6 + 2x^5 - 13x^4 + x^3 - 7x^2 + 11x + 17 = 0,$$

where is the process in all the development of symbolic logic that will determine them for us? If we ask for the hypercomplex numbers that have the characteristic equation

$$\xi^4 - 4\xi\eta^2 + 5\eta^3 = 2$$

who in all the mathematical world will send us the list? Or lists, we should say, for this one equation will determine more than one certain set of such hypercomplex numbers. Quaternions satisfy the equation

$$q^2 - 2Sq \cdot q + T^2q = 0,$$

but so do other hypercomplex numbers. Take the most definite function we can find, and what does it give? Not a class in extension, but certain properties that are found in some example we may produce, yet which may exist in an infinite number of other examples we have never thought of. Mathematics is full of discoveries of just such expansions of the extension of the notions that we have been using. It is a great advance, as Poincaré says, to find that we can bring two things, that is, two extensions under one name. As a simple case again, the notion of prime number is surely a definite thing, yet who knows how to ascertain whether 67989379012301 is a prime or not? The notion of simple group is definite, are there then simple groups of odd order, other than cyclic groups? When von Staudt called involutions on a line complex numbers, were they cases under the definition or not?

There is, as everyone knows, a vast difference between finding the value of a function for a given argument and finding the arguments that will satisfy a given function. In fact to meet the latter requirement mathematics has had to invent whole new extensions. Symbolic logic does not give us any assistance in this work of development nor any new methods. Its problem is only to criticise the character of the inferences involved in the process of development. It furnishes neither the major nor the minor premise but simply passes upon the validity of the transition from both to the conclusion. And all it furnishes in the propositional function is a sort of common invariant for many sets of classes. The classes may not exist in some senses, and may exist in others, just like the imaginary roots of an equation. But the propositional function does not point out the particular cases of members of the class it defines any more than it solves an algebraic equation. This is quite different from the explicit mathematical function,* like $\sin x$ or

$$\int_0^a \frac{\sin x dx}{x},$$

but analogous to the implicit function, like

$$\int \sqrt{y - x}\, dx, \text{ where } x^3y - xy^2 + y^5 = 4.$$

Types.

Chapter II of the Introduction develops the part of the Principia which differs most from the Principles, the *Doctrine of Types*. By this doctrine the authors hope they have resolved the paradoxes of the Principles as well as others that have been stated in discussions provoked by the original ones or by the theory of ensembles of Cantor.

The net result of the discussion seems to be that the predicate of a sentence is of the nature of a matrix or function symbol, and cannot serve as a subject for a sentence which has it also as predicate. Thus we might consider the statement "Triangle ABC is scalene." The functionality involved here, scalenity, cannot be put as the subject of this sentence. Scalenity is scalene, would be an absurdity. In this form at least, we might all admit that there is a sort of hierarchy of functions, if not of types. Of course one may talk of scalenity, but scalenity does not belong to the range that is itself scalene.

The *type* of a function is the class of objects that make it significant. That is to say in substituting values for x in φx some of them will give true propositions, some will give false propositions, some will give statements that are neither true nor false. That ensemble which produces with φ a proposition, true or false, is the type. The outcome is a little curious, as it leads to a reincarnated ghost of the buried *class*. The last paragraph of page 173 is interesting reading in connection with what has gone before. The difficulty seems to be of the same type as that which certain mathematicians find in recognizing any symbolic operator as an existent entity of the same character as the things it operates upon. Strictly speaking, of course there is a difference between $\frac{5}{1}$ and 5, at least till we come to see that the invariant properties we spoke of above are the same in the one case as in the other, and we identify the two. In just this same way we elevate ensembles in general into the region of functions, and then we can draw a distinction between φ as function or φ as argument.

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* Russell, *Revue Mét. et Morale*, vol. 13 (1905), pp. 906–917.

Whether we agree or disagree with the philosophy underlying the argument, we have gained one point more in our symbolism, that is, that we may make a function sign out of any symbol, simple or complex, and we may use any symbol as an argument for the proper function sign. In other words we may construct more and more complicated forms, and we may substitute for any single symbol or set of symbols, complex symbols. Incidentally we have a relativity theorem in mathematics in the doctrine of types. For no type is the bottom of all types. Types are only relative. An individual in today's discussion may be a function in tomorrow's. But do we resolve a contradiction by calling it absurd?

And it is difficult to see how the doctrine of types can be reconciled with many mathematical developments. Thus if we define the function φ by the differential equation $d\varphi x/dx = \varphi x$, we seem to have a case in opposition to the doctrine. For to define the derivative we must know an infinite set of values of the function φ, which is itself defined by means of the derivative. Apparently then we define a function in terms of the function itself. Another case more to the point possibly is the integral equation, and the integro-differential equation.

Relations and Descriptions.

The third chapter of the Introduction is devoted to descriptions, classes, and relations, under the title Incomplete Symbols. It is said that these can be defined only in their use. They are analogous to the symbols $\int$, ∇, $\sin^{-1}$, etc. However, it is pointed out that "the incomplete symbols are obedient to the same formal rules of identity as symbols which directly represent objects, so long as we consider the equivalence of the resulting variables (or constant) values of the propositional functions and not their identity. This consideration of the *identity* of propositions never enters into our formal reasoning." Under the limitations to the use of these symbols we find that, while x is always identical with x, yet the round square is not identical with the round square, for the reason there is no round square. This surely resembles the argument: Nothing is better than heaven, a shilling is better than nothing, therefore a shilling is better than heaven.

The descriptive symbol is $(\iota x)\varphi x$. It is used in precisely the same way as, and in fact differs little from, the symbol for a class. That is to say, it is a function sign, denoting the single object in the class. For example, the author of Waverly. This phrase does not point out the individual Scott, yet it identifies him. The legal John Doe does the same thing for a criminal who refuses to give his real name. These are, in the sense defined, function signs.

The relation is also a function, the function however having two arguments. There are many features common to these three incomplete symbols. The logic of relatives has also many developments that are not found in the others.

With regard to the three chapters of the Introduction, we desire to remark that they are in general somewhat difficult to apprehend as they now stand, for two reasons. First the distinctions drawn seem in many cases to be confused. It is sometimes difficult to ascertain whether the authors are using words in an every-day sense, in a philosophical sense, or in a purely technical sense. It is not always clear whether a symbol is under discussion, or the meaning of the symbol, or the use of the symbol. The introduction of many more good examples might have remedied this defect in style. In the second place, the statements do not seem to be thoroughly consistent. For example, it is not easy to decide what the authors mean by extensional. In the early part it seems to mean, as ordinarily in logic, the totality of individuals constituting an ensemble or collection. Later it seems to mean anything intellectual as distinguished from the emotional and the volitional. Then on page 196: "Propositions in which a function φ occurs may depend, for their truth-value, upon the particular function φ, or they may depend only upon the *extension* of φ. In the former case, we will call the proposition concerned an *intensional* function of φ; in the latter case an *extensional* function of φ." Also the apparent dread the authors exhibit towards the word *concept* seems to make the explanations often involved. The root of the whole difficulty seems to lie in an unconscious, or at least unstated, philosophical theory that a general notion is only symbolic and has no real existence of its own, but is existent only as it is manifested in some supposed concrete form. This notion seems to be the source of many of the peculiar interpretations forced upon the symbols. We may in most cases interpret them otherwise, as we have tried to point out; and as a calculus of logic, the system given here is very complete. We consider next the constructive features of the work.

Elements.

Substantially all that is meant by a *proposition* is to be found in the formal character of the two symbols p and $\sim p$. The latter is called the contradictory of p. It is of such a nature that the function $\sim$ is involutory, that is, $\sim\sim p$ is the same as p. If we understand then that we have a set of symbols p, q, r, or φa, φb, φc, and the like, with the duplicate set of their contradictories, we have the elements of the subject as analysed in this book. The descriptive symbol is substantially the same as we find exemplified mathematically in the symbol $+\sqrt{2}$, which in the present notation would be $(\iota x)\ (0<x, x^2=2)$. That is to say, we introduce into arithmetic the indefinite symbol x with the agreement that we will insert it in our number series, and that for x^2 we will always write 2. This is, in a more general case, like the Kronecker modular theory. It is along the line of the algebraicising of mathematics, as opposed to the arithmetising of mathematics. It is immaterial from this point of view whether the thing x exists or not. If in any sense it does exist then we may use a single symbol for it rather than the long form $(\iota x)\ (\varphi x)$. Practically the idea of class is of the same kind. A mathematical example is the definition of a class of algebraic numbers by an equation, as $x^4 + 5x^3 - 2x^2 + 3x - 13 = 0$. The idea of a symbolic class ought not to disturb a geometer, for he is used to imaginary points, circular points at infinity, and the like. We talk of these ideal things as if they existed, being careful that our phrases have meaning when we find the entities do exist in any sense. The same notion occurs again in the relative. A mathematical example is the equation of a curve. Thus $x^2 + y^2 = 25$ furnishes a relation, which finds as its proper representative the *curve* itself, while the pairs of arguments x, y furnish the points on the curve. The curve is a correlating agency for bringing together these pairs of arguments. Whenever there are entities which may be considered to be

represented by these symbols, then we use single letters for them and treat them in the old-fashioned manner of handling classes. This view of mathematics, and of propositions in general, we would prefer to call *functional,* rather then extensional, or intensional. It is surely mathematical, whether it is logical or not, and makes mathematics the fundamental basis of all reasoning, even more than the specialized interpretation called symbolic logic. On the philosophical side it seems to emphasize the statement that all reasoning in the last analysis is not about things but about relations between things, for every function expresses a relation between its argument and its value, and every relation may be referred to an ideal thing at least.

Combinations.

We must consider next the combinations that are actually built up out of these elements. The general development is given in Part I, which extends from page 89 to page 342, and is called Mathematical Logic. We desire to consider it apart from any meaning of a specific character that might be attached to the symbols. There is a single combination introduced, represented thus: $p \vee q$. Any two symbols may be joined in this manner. It is commutative, that is $p \vee q$ is the same as $q \vee p$. Special symbols are used for the combinations $\sim p \vee q$, which is written $p)q$, and the combination $\sim(\sim p \vee \sim q)$, which is written $p \cdot q$, the first being called implication, the latter logical product, while the basal combination is called disjunction. Any two of these symbols may be omitted. Indeed for ease of manipulation, it seems that to express everything in terms of the symbol $\cdot$ would be best. Thus for $p \vee q$ we write $\sim(\sim p \cdot \sim q)$, and for $\sim p \vee q$ we write $\sim(p \cdot \sim q)$. We agree further to certain permissible reductions or expansions. Thus, for example, we may write for p,

$$p \cdot p, \quad \text{or} \quad p \vee pq, \quad \text{or} \quad p \cdot q \vee p \cdot \sim q, \quad \text{or} \quad p \vee (p \cdot \sim p) \cdot q,$$

whatever q may be. In fact if we consider that in any product p and $\sim p$ are incompatible and such product may be dropped after the sign v, we arrive at one of the simple methods of handling this calculus. We agree further that if we have any expression φx, where x is variable, we may write the symbol $(\exists x)\varphi x$, and likewise if we have φx and φy(*9.1 and *9.11). Further we agree that in φx we may substitute for x any symbol a of the same type as x, and also that x may run over a given range of proper type (*9.14 and *9.13). Also we agree that in any expression φa, where a is a constant, we may consider φ by itself as a function, and vice versa (*9.15). We may now take any combination of symbols and build more complicated ones with the use of the v or the ●. We add, however, to the symbolism an expression for the range, thus: (x), meaning all values of x, and $(\exists x)$ meaning those values which are solutions. The name formal implication is given to the combination

$$(x)\,\varphi x)\,\psi x \text{ or } \varphi x)_x\,\psi x.$$

In using functions of two variables we introduce the abbreviations called relations. To indicate that x is a solution of φx we abbreviate thus: $x \,\epsilon\, \varphi \,|\, x$.

The formulas resulting from these few elements are very numerous, and no attempt will be made to go into them. It is from the results of these combinations however that the authors expect to produce other combinations which will have all the properties of numbers, series, etc.

It is obvious that, out of all the processes the mind goes through, others might have been selected as the fundamental ones. Whether a commutative combination is in the end more useful than one that is not commutative may be a question. We often must use a non-commutative product in mathematics, as "if p is *first* true, and then q is true, it follows r is true." The whole consideration of mathematical form* might from certain points of view be considered to be a prerequisite to the study of any kind of combination. In Whitehead's Universal Algebra this is partly in evidence.

We need to note further that the few modes of combination used here are really supplemented later by a free use of relational symbols, P, Q, R, ϵ, P_Δ, in fact by so many that when we remember the few combinations used here we wonder why these in particular should have been chosen to be represented by arbitrary signs. Indeed the number of arbitrary symbols which have to be memorized is so great in the book that one is willing to conclude that a more significant system could have been worked out. But taken as it is, there remains still the mathematical problem. Stated in brief it is this: given two operations by which from elements or marks new elements or marks are produced, namely, let us say, given that from p, q we construct

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* Cf. Kempe, *Trans. R. Soc. London*, vol. 177 (1886), p. 1–70; *Nature*, vol. 43 (1890), 156–162; *Proc. Lond. Math. Soc.*, vol. 26 (1894), 5–15.

$$s = \varphi p, \quad t = \psi(p, q), \quad \text{and} \quad u = \theta(p, q)$$

with the conditions or identities that

$$\varphi s = \varphi\varphi p = p, \quad \varphi\psi(p, q) = \theta(\varphi p, \varphi q) \quad \text{or} \quad \varphi\theta(p, q) = \psi(\varphi p, \varphi q),$$
$$\psi(p, q) = \psi(q, p), \quad \theta(p, q) = \theta(q, p).$$

We now have the properties of these combinations as combinations to consider. This problem is one of general algebra, universal algebra, or multiple algebra, according to the title preferred, and has at its base the very fundamental question as to what a combination is logically, psychologically, and otherwise; and what the operator φ, or θ, or ψ may be, what it does to the operand, what operators are derivable from it and how; further, what the result of the operation, say s, t, u above, is; how one may pass from the operand to the result, and reversely. These are elements that seem to be overlooked in the development as given in the book. Before one uses a calculus, in other words, he should investigate the laws of his calculus.

In the course of such investigation, it turns out that structural laws are very numerous. We may investigate laws that have been assigned purely arbitrarily, as for example those that actually have been so assigned in the study of multiple algebra. The reduction of all these divers arbitrary types of structure to a few simple forms is not possible, and the introduction of extra relational symbols merely furnishes a symbolism, but does not account for the forms nor does it show that they are deducible from the primitive forms laid down in this book as the basis of all reasoning. And if what is meant is that mathematical form consists of relations, then nothing is done beyond furnishing a mere name to a class of entities. Let us put it otherwise: to single out a few combinations as worthy of special signs, and to represent all others as relations, using letters, does not substantiate the claim that all terms have been defined in terms of the few combinations. The expressions for relations $\hat{x}\hat{y}\varphi(x, y)$ are combinations and the original $p \vee q$, $p \cdot q$, and $p)q$ can be so expressed; for each is not different from $\varphi(a, b)$ for a properly chosen φ, hence are cases of $\varphi(x, y)$, therefore define relations. To say that ultimately all logic is reducible to propositional functions, would then be the proper outcome. And logic becomes thus a branch of general algebra.

Prolegomena to Cardinal Arithmetic.

This constitutes Part II of the present book. Part III treating of Cardinal Arithmetic, Part IV of Ordinal Arithmetic, Part V of Series, are mentioned for the following volumes. The subjects treated in Part II are of high importance not only for cardinal arithmetic but for the ensemble theory. The divisions are Section A, Unit classes and couples; Section B, Sub-classes, Sub-relations, Relative types; Section C, One-many, One-one, Many-one relations; Section D, Selections; Section E, Inductive relations.

We find 1 defined here by the symbol

$$1 = \hat{\alpha}\,[(\exists x)\cdot \alpha = \iota\text{‘}x] \quad \text{Df.}$$

In words, 1 is the class of all unit classes, or since we have abolished classes per se, we will paraphrase this to read: 1 is a function satisfied by nothing but those functions which are true each in one case only. Or again, one is a property possessed by functions, namely, uniqueness of argument. For example, we speak of the author of Waverly, the President of the United States, the sin 30°, all these enable us to put the word *the* in evidence, and the *the-ness* in their character is that common property called 1. Whether this is a logical definition of the everyday 1 or not, what is accomplished is the construction of a symbol out of those already existing which defines the property of uniqueness. We agree to use 1 in place of the longer form $\hat{\alpha}[(\exists x)\cdot\alpha = \iota\text{‘}x]$. Likewise 0 is defined to be the function satisfied only by those functions which are never true. For example, () is not identical with itself, () is true when its contradictory is true, and such like. These are in no case true, and the property of their impossibility is the number represented by 0. The cardinal 2 is defined similarly as a function satisfied only by functions which define couples. If the couple defined were an ordered couple, then we define 2_r, the ordinal 2. For example, 2 is a property possessed by quadratic equations, when we confine our attention to the solutions they have. From the purely symbolic point of view we use 0 and 2 to abbreviate the forms

$$0 = \iota\text{‘}\Lambda \quad 2 = \hat{\alpha}[(\exists x, y)\cdot x \neq y\cdot \alpha = \iota\text{‘}x \vee \iota\text{‘}y] \quad \text{Df.}$$

Mathematically we have thus found that if the words one, none, two are used, we must have in mind the *uniqueness* of certain classes, the *impossibility* of certain classes, or the *dyad character* of certain classes. We may accept or reject this view of what the symbols mean, but practically we have placed 0, 1, 2, in the list of symbols which form the range of classes of classes. They belong to the second order symbols. The symbolic point of view is nearly the same as saying that we start with objects, these are entities of any order m. Then we make a set of tags to enable us to distinguish the objects without being concerned with their other qualities. These tags are $m + 1$ order symbols (classes). We then make a set of symbols to enable us to talk about the tags. This set of symbols is the set of cardinal numbers, and is of the $m + 2$ order in the process of symbolizing or abstracting.

We begin now to reach arithmetic. As one example of what it looks like, we will quote the theorem which is to prove later that $1 + 1 = 2$. It runs thus

$$*54.43 \vdash \,:\cdot\, \alpha, \beta \epsilon 1 .) : a \frown \beta = \Lambda \cdot \equiv \cdot \alpha \smile \beta \epsilon 2.$$

That is in English, if α and β are unit-classes with no common members, then their smallest superclass is a couplet.

In Section A we find also the ordinal 2_r, which does not differ from the class of alio-vids of C. S. Peirce; and also the ordinal $\dot{2}$, which does not differ from the class of vids. An ordinal 1 might be defined by $\dot{2} - 2_r$, which is the common subclass of vids and vids that are not alio-vids, that is the class of idem-vids. The connection with the cardinal 2 is in the fact that an alio-vid, that is, an asymmetric relation, must have two distinct terms. The relation here, being a vid, is between one object of thought and one other object of thought. Of course we are not far from the theory of quadrate algebras and matrices in general after we have arrived at this result.*

We pass over the next three sections, although they are of high importance, to Section E, which treats of generalized mathematical induction. The notion of *hereditary* class is defined, μ is a hereditary class with respect to the relation R if successors of μ's are μ's. For example if μ is the peerage, μ is hereditary with respect to the relation of father to eldest son. If μ is numbers greater than 100, μ is hereditary with respect to the relation of ν to $\nu + 1$. Mathematical induction is evidently included in this class of relations, and by means of it we pass from any finite integral case to any greater integral case, but finite. No such proof holds for any infinity. Thus we may say the binomial theorem is proved in this manner for *any* finite integral exponent, but not for *all* finite integral exponents. Indeed the word *all* here has no sense. We are led to consider powers of relations and the analysis of the field of a relation. This belongs to the general theory of operations.

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* C. S. Peirce, *Amer. Jour. Math.*, vol. 4 (1881), p.221.

The whole of the second Part really is mathematical logic of a little more specialized character than Part I, and this first volume could have properly been called a treatise on the mathematics of logic.

Summary.

In summary, the object that we have had in mind was to show that this first volume of the Principia is in reality an application of mathematical methods of definition and synthetic combination to the relationships between the abstract things that logic chooses to discuss. By means of a symbolism, which awkward as it is, is sufficiently comprehensive, a study is made of functions: as related to terms in propositions, and as shown in the particular forms of descriptions, classes, and relations. In one sense the highly ideal character of mathematical objects is made more evident. In another sense the real mathematical object, though already ideal, is sublimated still further into a logical object. The book is a culmination of the critical investigation of mathematical foundations of recent years, and will no doubt advance the systematization and mutual readjustment of mathematical treatments. It will assist in discovering tacit hypotheses, and in putting into formal shape the demonstration of many facts that have been brought to light by the intuition. If it eventually helps in any substantial manner to unify different theories and show their common features it will do enough.*

But while we may admit that it has perhaps placed the fundamental principles of the theory of ranges in a more definite form, and has done something for the theory of relations, we insist that the other great theories of mathematics are barely touched upon if, indeed, at all. These, we pointed out, were the theory of structure and form, the theory of invariance, the theory of functions as functions, the theory of inversions. That these can receive a general treatment we do not doubt, inasmuch as some of them are receiving such development. In logistic then we find only a very definite *branch* of mathematics, and in this volume we have the most complete treatment of logistic that exists. The question that many have asked naturally "How far does it assist in building up synthetic systems of mathematics" is easily answered. It reaches arithmetic only after one volume of 666 pages. We would not expect the complete treatise then to furnish much that would be of a synthetic nature. Indeed that would be as unreasonable as to expect to build Eiffel towers and Eads bridges from a study of postulates and axioms for the foundation of geometry. While design rests upon these things in a sense, design antedates them just as language antedates grammar. It is not fair to the book or its aim to assert that it does nothing synthetic. Its problem is philosophical and analytical. It does enough if it shows us what are the characteristic features of reasoning and generalizes the types of reasoning. In this respect it is scientific as well as philosophical. It examines the rules of the great mathematical game. But it does not play the game nor undertake to teach its strategy.

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* M. Winter, *Revue Mét. et Morale*, vol. 15 (1907), pp. 186–216. E. Borel, ibid., pp. 273–383. H. Poincaré, ibid., vol. 17 (1909), pp. 451–482, 620–653. H. Dufumier, *Bull. des Sciences Math.*, vol. 35 (1911), pp. 213–221.

JAMES BYRNIE SHAW.

DIFFERENTIAL GEOMETRY.

Vorlesungen über Differentialgeometrie. Von LUIGI BIANCHI. Autorisierte Deutsche Übersetzung von MAX LUKAT. Zweite, vermehrte und verbesserte Auflage. Leipzig und Berlin, B. G. Teubner, 1910. xvi + 721 pp.

IN 1899 Guichard announced (*Comptes Rendus* 128, page 232) without proof the following theorems:

I. Let M be a point of a quadric of revolution Q whose axis is of length $2a$, F_1 and F_2 being the foci of Q and φ_1, φ_2 the points symmetric to F_1, F_2 with respect to the tangent plane to Q at M; let S be a surface applicable to Q; as S is applied to Q the points F_1, F_2, φ_1, φ_2 invariably fixed with respect to the corresponding tangent plane to Q take positions which are denoted by F_1', F_2', φ_1', φ_2'; the loci of these respective points describe four surfaces for each of which the mean curvature is equal to $1/a$.

II. When the quadric Q is a paraboloid of revolution the loci of the points F_1' and φ_1' are minimal surfaces.

The announcement of these theorems marked the beginning of a new epoch in the theory of the deformation of surfaces. During the following years many eminent geometers attacked problems suggested by them, and in particular the deformation of the general quadric. This problem was solved in the admirable memoirs of Bianchi and Guichard which were jointly crowned by the French Academy in 1908. It is in the exposition of this theory and allied questions that the second edition of Bianchi-Lukat's Vorlesungen differs from the first, and is "vermehrt und verbessert." For the sake of brevity we shall refer to the two editions as (I) and (II).

Chapters 21 and 22 of (I) which deal respectively with N-dimensional space of constant curvature and hypersurfaces in spaces of constant curvature are omitted from (II), in order to make room for the theory of deformation of quadrics. It is to be regretted that this abridgment was necessary and that the treatise did not appear in two volumes so that this important theory, as well as topics subsequently referred to, could have been included. In view of this omission practically all of the quadratic differential forms which appear in the book involve only two variables and so there is no necessity of developing the theory for n variables as in (I). One finds accordingly that in (II) Chapter 2 is given over to the theory of quadratic differential forms in two variables. This makes the treatment simpler in places, but we cannot see that there was any gain made by the change. There is introduced an algebraic proof of the theorem: Two binary quadratic differential forms with the same constant curvature are always equivalent, and the transformation equations involve three arbitrary constants.

The other changes in the first part of the book are a few additions to the theory of deformation in general. Those double families of lines upon any surface S which become asymptotic lines on one of the deforms of S are called *virtual asymptotic lines*. The determination of these families of curves requires the solution of two simultaneous partial differential equations of the second order. Applying the theory of the Picard method of successive approximations, it is shown that any two intersecting analytic curves C and C' on an analytic surface S determine uniquely a double family of virtual asymptotic lines on S to which these curves belong, one to each family, and the deform of S on which these curves are asymptotic lines is analytic. This theory is applied in more detail to pseudospherical surfaces in Chapter 17, but it is similar to the treatment of this particular case given in Chapter 17 of (I).

At the end of Chapter 8 one finds a theorem due to Chieffi which is of great service in the subsequent discussion. If S is a non-ruled surface applicable to a ruled surface R, to the straight lines on R correspond a family of geodesics on S, called g. Let a be an asymptotic line of either system on S and draw the tangents to the curves g at points a, thus forming a ruled surface R_1. The theorem of Chieffi is that R_1 is applicable to S in such a way that a remains rigid during the deformation.

In Chapter 17 we begin to find traces of the results growing out of the theorems of Guichard. After developing the theory of Bäcklund transformations of pseudospherical surfaces as found in (I) (in places the treatment is slightly different), the author considers the real transformations obtained by combining conjugate imaginary ones, and more particularly the result of taking two opposite transformations B_σ and $B_{-\sigma}$ in applying the "theorem of permutability" (Vertauschbarkeitssatz). Let two such transformations be applied to a pseudospherical surface S, and let S_1 and S_2 be the resulting surfaces and S' the fourth surface which is the transform of these two; the normals to S_1 and S_2 meet in a point M_0 equidistant from these surfaces and likewise the normals to S and S' in a point M_0'. The two surfaces S_0 and S_0', the respective loci of M_0 and M_0', are applicable to the same surface of revolution and are complementary to one another. However, S_0 is not a pseudospherical surface as stated on page 483. According as σ is zero, real and different from zero, or pure imaginary, the meridian curve of the surface of revolution is different, when the latter surface is real. But for all three types of σ the surface S_0 is applicable to the surface of revolution of an imaginary ellipse. Thus one is brought incidentally into contact with the problem of deformation of quadrics.

With the exception of the first five pages the material in

Chapter 18 does not appear in (I). By means of a change of variables the equations of the preceding chapter are given a form which define a Bäcklund transformation of spherical surfaces, that is, surfaces applicable to the sphere, with the difference that all of the first transforms are imaginary. However, these may be combined into real transformations as described in the following theorem: If S_1 is an imaginary Bäcklund transform of a spherical surface S by means of a transformation B_{σ_1}, $(\sigma_1 = a + ib)$, and S_2 is the transform of S by $B_{\bar{\sigma}_1}$, $(\bar{\sigma}_1 = a - ib)$, the fourth surface S' given by the "theorem of permutability" is a real spherical surface. When $b = 0$, the normals to S and S' meet in a point M_0, and the normals to S_1 and S_2 in a point M_0'; the surfaces S_0 and S_0', the loci of these points, are complementary surfaces and are applicable to the same prolate ellipsoid of revolution. When $b = \frac{1}{2}\pi$, S_0 and S_0' are applicable to a hyperboloid of revolution of two sheets.

The second part of Chapter 18 is given over to the proof of the two theorems of Guichard stated at the beginning of this review, and converses of these theorems. In preparation for this discussion the author considers congruences of spheres, whose surface of centers S_0 undergoes a deformation, and also congruences of lines invariably fixed to a surface which is being deformed. If one of the sheets of the envelope of spheres has constant mean curvature as S_0 is deformed, the same is true of the other sheet. If two sheets are minimal surfaces, S_0 is applicable to a paraboloid of revolution P, the radius of a sphere being the distance between the corresponding point on P and the focus. If the mean curvature of the two sheets is different from zero and constant, S_0 is applicable to a prolate ellipsoid of revolution E or a two sheeted hyperboloid of revolution H, and the radius of a sphere is the distance between the corresponding point on E or H and a focus.

The Bäcklund transformations of a pseudospherical or a spherical surface S may be looked upon as transforming the surface elements of S, that is the points of S (called the *centers* of the elements) and small portions of the tangent planes about the points, into ∞^3 surface elements in space which can be grouped so as to be the surface elements of ∞^1 surfaces, the transforms of S. This point of view is not peculiar to these transformations. As a matter of fact it is thus that the author sets up the transformations of surfaces applicable to any quadric in the following very simple and remarkable manner: Let Q be any quadric and Q_1 a confocal quadric; let a surface element f have for center F, a point of Q, and for its plane π the tangent plane to Q at F; this plane π meets Q_1 in a curve C; take a point F_1 of C as the center of the surface element f_1 whose plane π_1 is determined by the line FF_1 and one of the rulings of Q_1 through F_1, it being understood that rulings of the same family are used in setting up the surface element for each point of Q; the plane π_1 is tangent to Q_1 at some point of the ruling. The fundamental theorem is the following:

THEOREM (A): *If Q is deformed in any manner into a surface S and in the deformation to each element f there are invariably coupled ∞^1 elements f_1, obtained by varying the point F_1 of C for each point F, these ∞^3 surface-elements can be correlated into a unique family of ∞^1 surfaces S_1, each of which is applicable to S and to Q.*

This transformation which converts S into a surface S_1 is called a transformation B_k, where the subscript k is a constant determining Q_1 among the quadrics confocal to Q. The proof of this theorem for the case when Q and Q_1 are hyperbolic paraboloids occupies, with certain allied problems, the whole of Chapter 19. The processes are in keeping with the general treatment of problems of differential geometry in the former chapters of the treatise and as a matter of fact it is essentially that followed by Bianchi in his prize memoir. The coordinates of Q and Q_1 are given in explicit form in terms of parameters referring to their generators. By straightforward steps the relations between the surface elements f and f_1 are established and it is shown that the correlation of the ∞^3 elements f_1, arising from the ∞^2 elements determined by a surface S, into ∞^1 surfaces S_1 requires the integration of an equation of the Riccati type. When the linear element of S_1 is calculated, it is found that it is not of the same form as that of S; consequently S and S_1 are not applicable in such a way that the centers of corresponding surface elements correspond in applicability, if at all. Since Q and Q_1 are confocal paraboloids of the same family, the orthogonal trajectories of the family establish a correspondence between Q and Q_1, namely corresponding points lie on the same trajectory. This is the transformation of Ivory of the one quadric into the other; it plays a very important role in this theory, as we shall see. Let S be any deform of Q and S_1 one of the surfaces defined by the correlation of the displaced surface elements f_1. If M_1 on S_1 corresponds to F_1 on Q_1, and $\overline{M}$ on S to the point F on Q which is the Ivory transform of F_1, the surfaces S and S_1 are applicable with the points $\overline{M}$ and M_1 in correspondence. The proof of this result covers ten pages most of which involves extensive calculation, so that at times the reader's interest lags, but throughout the whole treatment the geometrical ideas are extremely clear. There is no indication of the reason for hitting upon this correspondence of Ivory as the one leading to applicability. Later in the chapter certain properties of the Ivory transformation are derived as a means to the proof of the theorem: If S_1 is the transform of S by a B_k, then S is a transform of S_1 by the same B_k; these properties may have led to the suggestion of the trial of the transformation as a good guess, but the reader is left in the dark. The development of the results throughout the chapter is simplified and made more interesting by associating with the surfaces S the ruled surfaces R, applicable to S, as derived by means of the Chieffi theorem. Among these results one finds this pretty theorem: If the quadric Q rolls upon an applicable ruled surface R, each of the congruences Γ, $\overline{\Gamma}$ generated by the lines of the first and second system on a confocal quadric Q_1 may be arranged in a unique manner into a simple infinity of ruled surfaces R_1, each of which is applicable to Q and to R; moreover, the applicability of R_1 on R is accomplished by a continuous deformation. The surfaces R and R_1 of the foregoing theorem possess the following property in common with the non-ruled deforms of Q:

THEOREM (B): *The lines MM_1 joining corresponding points on a surface S and a surface S_1, resulting from S by a transformation B_k, constitute a W-congruence of which S and S_1 are the focal surfaces.*

The remainder of Chapter 19 is given over to the proof of Theorem (B) for the case where Q is a hyperbolic paraboloid. The method is much simplified by use of the results for the ruled surfaces R and R_1.

In Chapter 20 Theorems (A) and (B) are established for the case where Q and Q_1 are confocal hyperboloids of one sheet. The method followed is similar to that of the preceding chapter with very little abridgment. In fact the formulas differ only in form, and if anything they are more involved in the present case. Consequently the geometrical results are practically the same in the two chapters. From a historical point of view this fact is very interesting, since the earlier methods which were fruitful in the study of the deforms of paraboloids could not be applied in the case of central quadrics. It should be mentioned, however, that the exceptional case of hyperboloids of revolution leads to a theory of transformation of Bertrand curves analogous to the transformation of Razzaboni.

From the manner of construction of the surface elements f_1 it follows that if Q is real these elements are real only when Q_1 has negative curvature. Hence whatever be the type of Q, if only real, the surfaces S applicable to it admit real transforms S_1. However, the Ivory transformation makes real points on Q_1 correspond to imaginary points on Q, when the latter is not a hyperbolic paraboloid or a hyperboloid of one sheet. In this case the applicability of S and S_1 is *ideal.* These cases are treated in Chapter 21. By a slight change of notation the formulas of the two former chapters are used to establish Theorems (A) and (B) for the elliptic paraboloid, the hyperboloid of two sheets, and the ellipsoid. A novel converse result is that any real surface applicable in the ideal sense to a quadric of any of these three types admits of ∞^1 transforms S_1 applicable to the same quadric in the real sense. In a like manner there are real surfaces applicable in the ideal sense to the hyperbolic paraboloid and the hyperboloid of one sheet; such a surface is transformable by a B_k into a real surface applicable to the same quadric in the ideal sense. The discussion of these transformations closes with the following "theorem of permutability:"

If S is a surface applicable to a quadric Q, and S_1 and S_2 are two transforms of S by means of transformations B_{k_1} and B_{k_2}, where k_1 and k_2 are different constants, there is a fourth surface S', also applicable to Q, which arises from S_1 and S_2 by transformations B_{k_2}' and B_{k_1}', respectively.

For the proof of this theorem the reader is referred to the third volume of the second Italian edition. Here and in the memoir presented to the French Academy one will find also a treatment of surfaces applicable to the imaginary quadrics. One is inclined to feel that owing to the similarity of treatment of the several cases and in view of the existence of these details elsewhere much of Chapter 20 and part of Chapter 21 should have been omitted, and the space thus gained should have been used for other material appearing in the second Italian edition which sets forth new ideas. For example, there might have been included the treatment of surfaces with plane or spherical lines of curvature, or an exposition of the Weingarten method of deformation. If the latter had been developed in the manner of the second Italian edition, the ideas and formulas of moving axes would have been introduced, although rather late. We feel that the omission of this operator from the German editions is a serious one. It is to be greatly regretted that both the German and Italian second editions were not rewritten to such an extent that the moving axes could have been used to advantage.

The last three chapters deal with triply orthogonal systems of surfaces. The contents and treatment in the first two of these chapters are essentially the same as in the first edition. By far the greater part of the second chapter deals with confocal quadrics and geodesics on quadrics. This material could have been introduced into Chapter 6 and it would have served as a fine illustration of the general theory there set forth. We have always felt that this should have been done in the first editions. In fact we cannot see the reason for concealing this subject matter in a chapter which a beginner is not likely to read. The last chapter of (I) deals with Lamé families of pseudospherical surfaces. In the last chapter of (II) the treatment of spherical surfaces as well is included, together with that of certain surfaces applicable to quadrics.

We were pleased to find a fairly complete index in addition to a table of contents, which appears in (I) alone. The translation is well done, there are very few typographical errors and the bookmaking is up to the standard of the Teubners.

LUTHER PFAHLER EISENHART.

NOTES.

THE April number (volume 13, number 2) of the *Transactions of the American Mathematical Society* contains the following papers: "A generalization of Weierstrass's preparation theorem for a power series in several variables," by G. A. BLISS; "An existence theorem for periodic solutions," by W. D. MACMILLAN; "A condition that a function in a projective space be rational," by W. F. OSGOOD; "A method of proving certain theorems relating to rational functions which are adjoint to an algebraic equation for a given value of the independent variable," by J. C. FIELDS: "The dependence of focal points upon curvature for problems of the calculus of variations in space," by M. B. WHITE; "Orthocentric properties of the plane directed n-line," by J. E. HODGSON; "Certain singularities of point transformations in space of three dimensions," by S. E. URNER; "Multiple correspondences determined by the rational plane quintic curve," by J. R. CONNER.

THE April number (volume 34, number 2) of the *American Journal of Mathematics* contains: "The groups of birational transformations of algebraic curves of genus 5," by J. V. McKELVEY; "The general theory of linear q-difference equations," by R. D. CARMICHAEL; "Concerning linear projective order," by A. R. SCHWEITZER; "On certain expansions of elliptic, hyperelliptic, and related periodic functions," by F. R. MOULTON; "On cubic birational space transformations," by H. P. HUDSON; "On the definition of multiplication of irrational numbers," by OSWALD VEBLEN.

THE Library of Congress will have ready for distribution in July the first of a series of catalogue cards forming the beginning of a dictionary catalogue of all articles in the Encyklopädie der mathematischen Wissenschaften and the Encyclopédie des Sciences mathématiques. The material for all parts of the two Encyclopedias thus far issued has already been supplied by Brown University, which will continue to supply the material as future parts appear.

AT the meeting of the London mathematical society held on March 14 the following paper was read: By G. T. BENNETT, "The cubic surface as a degenerate quartic."

THE date of the Fifth international congress of mathematicians at Cambridge, England, is August 22–28. Series of lectures are announced by E. BOREL, E. W. BROWN, A. KNESER, E. G. H. LANDAU, J. LARMOR, and Sir W. WHITE. The American members of the organizing committee are Professors M. Bôcher, J. C. Fields, E. H. Moore, F. R. Moulton, B. O. Peirce, J. Pierpont, and A. G. Webster. Requests for information regarding the congress should be addressed to the general secretary of the organizing committee, Professor E. W. Hobson, Christ's College, Cambridge.

THE academy of sciences of Paris announces that the grand prize (3000 frs.) will be awarded in 1914 for the solution of the following problem. "To perfect the theory of functions of one variable which can be represented by trigonometric series of several arguments which are linear functions of the variable. The academy will be pleased to consider important applications to mathematical physics and to celestial mechanics." The following problem is proposed for the Fourneyron prize (1000 frs.) in 1914. "Theoretical and experimental study of turbines."

THE royal academy of sciences of Belgium proposes the following prize problems for 1913: "An important contribution to the infinitesimal geometry of curved surfaces" (800 frs.). "Summarize the theory of systems of space cubic curves and make a new contribution to the theory" (800 frs.). Competing memoirs should be sent to the secretary before August 1, 1913.

THE first congress of Russian mathematicians, including university professors and teachers in the secondary schools was held at St. Petersburg, January 6–16, 1912, in which more than 1200 persons participated. The president of the congress, Professor A. VASSILIEF, delivered an able address "On instruction in mathematics and philosophy in the secondary schools." Extracts of the papers will appear in an early number of *Enseignement Mathématique*.

THE following university courses in mathematics are offered during the summer semester of 1912.

UNIVERSITY OF BERLIN.—By Professor H. A. SCHWARZ:

Space curves and surfaces, four hours; Analytic functions, four hours; Hypergeometric series, two hours; Colloquium, two hours; Seminar, two hours. — By Professor G. FROBENIUS: Determinants, four hours; Seminar, two hours. — By Professor F. SCHOTTKY: Quadric surfaces, four hours; Riemann's theory, four hours; Seminar, two hours. — By Professor R. LEHMANN-FILHÉS: Mechanics, four hours; with exercises, one hour. — By Professor G. HETTNER: Introduction to ordinary differential equations, two hours. — By Professor J. KNOBLAUCH: Applications of elliptic functions, four hours; Integral calculus, four hours; Selected chapters of analytic geometry, one hour. — By Professor I. SCHUR: Elliptic functions, four hours; Differential calculus, four hours; with exercises, one hour.

UNIVERSITY OF BONN. — By Professor E. STUDY: Differential and integral calculus, four hours; with exercises, one hour; Quaternions, two hours; Seminar, two hours. — By Professor F. LONDON: Descriptive geometry, four hours; Analytic geometry, II, three hours; with exercises, one hour. — By Dr. J. O. MÜLLER: Theory of numbers, three hours.

UNIVERSITY OF GÖTTINGEN.—By Professor F. KLEIN: Analytic geometry, four hours; Seminar, two hours.—By Professor D. HILBERT: Ordinary differential equations, four hours; Mathematical foundation of physics, four hours.— By Professor C. RUNGE: Mechanics, II, six hours; Seminar, two hours.—By Professor E. LANDAU: Algebraic numbers and ideals, four hours; The Galois theory of equations, two hours; Mathematical games and recreations, one hour; Seminar, two hours.—By Professor F. BERNSTEIN: Theory of probabilities, three hours; Mathematics of insurance, two hours; Seminar, two hours.—By Dr. O. TOEPLITZ: Differential and integral calculus, with exercises, six hours; Theory of an infinite number of variables, two hours.—By Dr. A. HAAR: Celestial mechanics, two hours.—By Dr. E. WEYL: Curves and surfaces, four hours; Theory of real variables, two hours; Exercises in analytic geometry, two hours.—By Dr. C. v. KÁRMÁN: Applications of the calculus of variations, three hours.—By Dr. R. SCHIMMACH: The international commission on mathematical instruction, two hours.

THE following university courses in mathematics are announced for the academic year 1912–1913.

COLUMBIA UNIVERSITY.—By Professor C. J. KEYSER: Modern theories in geometry, three hours; History and significance of central mathematical concepts, three hours.—By Professor T. S. FISKE: Introduction to the theory of functions of a real variable, three hours; Functions defined by linear differential equations, three hours.—By Professor F. N. COLE: Introduction to the theory of functions, three hours; Theory of plane curves, three hours.—By Professor JAMES MACLAY: Theory of numbers, three hours, first half-year; Differential equations, three hours, second half-year.—By Professor D. E. SMITH: History of mathematics, three hours.—By Professor EDWARD KASNER: Integral equations, two hours; Seminar in differential geometry, three hours.—By Professor W. B. FITE: Calculus of variations, three hours.—By Professor H. E. HAWKES: Modern higher algebra, three hours, second half-year.—By Dr. H. W. REDDICK: Differential equations, three hours, first half-year.—By Dr. N. J. LENNES: Projective geometry, three hours.

The mathematical colloquium will meet at intervals of about two weeks.

CORNELL UNIVERSITY.—By Professor J. McMAHON: Theory of probabilities, two hours; Vector analysis, two hours.—By Professor J. I. HUTCHINSON: Linear differential equations, two hours; Elliptic integrals, two hours (first term).—By Professor V. SNYDER: Analytic geometry of space, three hours (second term).—By Professor F. R. SHARPE: Introduction to mathematical physics, three hours.—By Professor W. B. CARVER: Theory of equations, two hours; Descriptive geometry, three hours (first term).—By Professor A. RANUM: Differential geometry, two hours.—By Professor D. C. GILLESPIE: Principles of mechanics, three hours.—By Dr. C. F. CRAIG: Elementary differential equations, two hours.—By Dr. F. W. OWENS: Projective geometry, three hours.—By Dr. J. V. McKELVEY: Algebraic curves, three hours.—By Dr. L. L. SILVERMAN: Advanced calculus, three hours.—By Dr. W. A. HURWITZ: Theory of functions of a complex variable, three hours.

THE following doctorates in mathematics were conferred by the French universities during the academic year 1910–1911:

Paris, Doctorat d'Etat.

Caubet, P. "Etude des principales inégalités du mouvement de la lune qui dépendent de l'inclinaison."

Chazy, J. "Sur les équations différentielles du troisième ordre et d'ordre supérieur dont l'intégrale générale a ses points critiques fixes."

Sire, J. "Sur les fonctions entières de deux variables d'ordre apparent total fini."

Chatelet, A. "Sur certains ensembles de tableaux et leur application à la théorie des nombres."

Gau, E. "Sur l'intégration des équations aux dérivées partielles du second ordre par la méthode de M. Darboux."

Annycke, Th. "Contribution à l'étude thermomechanique des tiges et des plaques."

Garnier, R. "Sur les équations différentielles du troisième ordre dont l'intégrale générale est uniforme et sur une classe d'équations nouvelles d'ordre supérieur dont l'intégrale générale a ses points critiques fixes."

Janiszewski, S. "Sur les continus irréductibles entre deux points."

Villat, H. "Sur la résistance des fluides."

Doctorat d'Université.

Gramont, A. de. "Essai d'aérodynamique du plan."

Lemaire, P. "Théorie des compas gyroscopiques."

Nancy, Doctorat d'Etat.

Arnoult, J. "Sur le mouvement d'un fil dans l'espace."

Lille, Doctorat d'Etat.

Barré, E. "Sur une classe de solutions des équations indéfinies de l'équilibre d'élasticité," et "Application de la géométrie cinématique à la théorie des surfaces engendrées par une courbe variable."

Professor G. B. Guccia, of the University of Palermo, has been elected a national member of the Royal Institute of Venice.

Professor G. Darboux, of the University of Paris, has been elected an honorary member of the Royal Irish academy.

DR. L. N. G. FILON has been appointed professor of applied mathematics at the University of London.

THE mathematical prize of the Prince Jablonowski society for 1911 has been awarded to Professor ERNST NEUMANN.

PROFESSOR W. KUTTA, of the technical high school at Aachen, has been appointed full professor of mathematics at the technical high school at Stuttgart.

PROFESSOR G. D. BIRKHOFF, of Princeton University, has accepted a call to an assistant professorship of mathematics at Harvard University.

AT the University of Missouri Dr. LOUIS INGOLD has been promoted to an assistant professorship of mathematics.

AT Columbia University Dr. L. P. SICELOFF has been promoted to an assistant professorship of mathematics. Mr. C. A. FISCHER has been appointed instructor in mathematics.

MR. R. W. BURGESS has been appointed assistant in mathematics at Cornell University.

AT Dartmouth College Dr. E. G. BELL, of Purdue University, has been appointed assistant professor of mathematics. Drs. J. E. ROWE, of Haverford College, and F. M. MORGAN, of Cornell University, have been appointed instructors in mathematics.

PROFESSOR J. AMSLER-LAFFON, inventor of the polar planimeter, died at Schaffhausen January 3, at the age of 89 years.

DR. EMIL LEMOINE, recipient of the Francoeur prize for several years, died at Paris, February 21, at the age of 72 years.

PROFESSOR OSBORNE REYNOLDS, of Owens College (now the University of Manchester) died February 21, at the age of 70 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Berry (W. J.). Differential equations of the first species. New York, Van Nostrand, 1912. 12mo. Cloth.

Bosmans (H.). Notes sur l'arithmétique de Simon Stevin. Louvain, Ceuterick, 1911.

Broll (G.). Mehrdeutige Verwandtschaften zwischen Punktfeldern, insbesondere solche, welche durch Raumkurven veranlasst werden. (Diss.) Breslau, 1911. 8vo. 86 pp.

Bücher, Neue, über Naturwissenschaften und Mathematik. Die Neuigkeiten des deutschen Buchhandels, nach Wissenschaften geordnet. Mitgeteilt Winter 1911–1912. Leipzig, Hinrichs, 1912. 8vo. pp. 69–92. M. 0.30

Cajori (F.). Notes on the history of geometry and algebra. Boston, Heath, 1911. 8vo. 25 pp. Paper. $0.10

Czuber (E.). Vorlesungen über Differential- und Integral rechnung. 1ter Band. 3te, sorgfaltig durchgesehene Auflage. Leipzig, Teubner, 1912. 8vo. 14+605 pp. Cloth. M. 12.00

Engelhardt (P.). Ueber die graphische und näherungsweise numerische Auflösung von Gleichungen 2ten und 3ten Grades mit einer Unbekannten im Unterrichte an der Mittelschule. (Progr.) Würzburg, 1911. 8vo. 33 pp.

Foerster (O.). Ueber Cassinische Kurven auf der Pseudosphäre. (Diss.) Münster, 1911.

Grandjean (K.). Ueber die mit einer Schläflischen Doppelsechs zusammenhangende Fläche F^3. (Diss.) Strassburg, 1911.

Heiberg (J. L.). Naturwissenschaften und Mathematik im klassischen Altertum. (Aus Natur und Geisteswelt. Nr. 370.) Leipzig, Teubner, 1912. 8vo. 4+102 pp. Cloth. M. 1.25

Joachimi (O.). Ueber Kurven bei denen die beiden Krümmungen durch eine quadratische Beziehung verknupft sind. (Diss.) Munster, 1911.

Kleber (A.). Ueber einige mehrdeutige Verwandtschaften zweier Ebenen. (Diss.) Rostock, 1911.

Kleeberg (R.). Ueber die Diskriminantenflächen der Gleichungen $A \cos x + B \sin 2x + C \cos 2x + \mathrm{D} \cos 3x = 0$. (Diss.) München, 1911. 8vo. 87 pp.

Kraft (K.). Das Normalenproblem an Kurven und Flächen zweiter Ordnung in den endlichen Raumformen. (Diss.) Munster, 1911.

Löwenherz (A.). Die Frenet'schen Formeln im R_{n+1}. (Diss.) Königsberg, 1911. 8vo. 73 pp.

Mohrmann (H.). Normalflachen und projektive Gruppen. (Habilitationsschrift.) Karlsruhe, 1911. 4to. 30 pp.

Mühe (A.). Beweis des Fermatschen Prinzips. Berlin, Reimer, 1911. 8vo. 3 pp. M. 0.30

Oster (E.). Zentralperspektive, stereographische Projektion und quadratische Binarformen. (Diss.) Strassburg, 1911.

Perl (E.). Untersuchungen über Differenzenkoeffizienten erster und zweiter Art, insbesondere über ihren Zusammenhang mit verwandten Grössen. (Diss.) Königsberg, 1911. 8vo. 126 pp.

Ratschläge für die Studierenden der Mathematik und Physik an der Universität Jena. 3te Auflage. Jena, 1912.

Rogers (R. A. P.). See Salmon (G.).

Salmon (G.). A treatise on the analytic geometry of three dimensions. Revised by R. A. P. Rogers. 5th edition. In two volumes. Volume 1. New York, Longmans, 1912. 8vo. 22+470 pp. $3.00

Schoenflies (A.). Ueber die allgemeinen Beziehungen der Mathematik zur Kultur und Erkenntnis. (Antrittsrede.) Frankfurt, Auffarth, 1912.

Stelzner (O.). Mathematik und Naturwissenschaften an den neuhumanistischen Schulen, unter Einwirkung von Gesner, Ernesti, Heine und Wolf. Ein Beitrag zur Geschichte des mathematisch-naturwissenschaftlichen Unterrichts. (Diss.) Leipzig, 1911.

Stoffaës (Abbé E.). Cours de mathématiques supérieures. 3e édition entièrement refondue. Tome 2: Courbes et surfaces. Equations différentielles. Paris, Gauthier-Villars, 1911. 8vo. 365 pp. Fr. 10.00

Sturm (A.). Geschichte der Mathematik bis zum Ausgange des 18ten Jahrhunderts. (Sammlung Göschen, Nr. 226.) 2te, verbesserte Auflage. Leipzig, Göschen, 1911. 8vo. 155 pp. Cloth. M. 0.80

Thue (A.). Einige Bemerkungen über die Gleichung $Ax^3 + By^3 = Cz^3$. (Videnskapsselskapets skrifter.) Utgift for Fridtjof Nansens fond. Kristiania, Dybwad, 1911. 8vo. 7 pp. M. 0.50

——. Ueber einige in ganzen Zahlen x und y unmögliche Gleichungen $F(x, y) = 0$. Utgift for Fridtjof Nansens fond. Kristiania, Dybwad, 1911. 8vo. 7 pp. M. 0.50

Toschi di Fagnano (G. C. De'). Opere matematiche, pubblicate sotto gli auspici della società italiana per il progresso delle scienze dai soci V. Volterra, G. Loria, D. Gambioli. Volumi I–II, che contengono la materia dei tomi I e II delle Produzioni matematiche. Milano, Albrighi, 1911. 8vo. 9+474 pp; 9+471 pp.

Umfahrer (J.). Beweis der Richtigkeit des "Grossen Fermatschen Satzes." München, Eher, 1912. 8vo. 16 pp. M. 0.50

II. ELEMENTARY MATHEMATICS.

Amaldi (U.). See Enriques (F.).

Barnard (S.) and Child (J. M.). A new algebra. Volume 2, containing parts 4–6. With answers. London, Macmillan, 1912. 8vo. 4s.

Belikow (A.) und Nathing (A.). Lehrbuch der Algebra. 2ter Teil, bearbeitet von A. Nathing. 3te Auflage. St. Petersburg, Eggers, 1911. 8vo. 3+179 pp. M. 3.00

Berichte über den mathematischen Unterricht in Österreich. 10tes Heft, von O. Simony, E. Kobald, A. Mikuta, und K. Reich. Wien, Hölder, 1911. 8vo. 39 pp. M. 1.20

Bos (H.). Géométrie élémentaire. 22e édition. Paris, Hachette, 1912. 16 mo. 288 pp. Fr. 2.00

Child (J. M.). See Barnard (S.).

Converse (H. A.). See Durell (F.).

Durell (F.). Plane and spherical trigonometry. With chapters on plane surveying by H. A. Converse. Also logarithmic and trigonometric tables. New York, Merrill, 1912. 8vo. 295+32 pp. $1.40

——. School algebra. New York, Merrill, 1911. 12mo. 507 pp. $1.10

Enriques (F.) e Amaldi (U.). Elementi di geometria, 4a edizione ridotta. Bologna, Zanichelli, 1911. 16mo. 13+351 pp. L. 2.50

——. Elementi di geometria, ad uso delle scuole secondarie superiori. 5a edizione. Bologna, Zanichelli, 1911. 16mo. 11+623 pp. L. 4.50

Fergusson (J. C.). The percentage unit of angular measurement, with logarithms. New York, Longmans, 1912. 67+666 pp. $20.00

Gebhardt. Der mathematische und physikalische Unterricht auf dem Vitzthumschen Gymnasium zu Dresden während des Zeitraumes 1861 bis 1911. Leipzig, Teubner, 1911.

Geometria, La, nel catasto italiano. (VI congresso nazionale fra geometri, agronomi, e periti agrimensori italiani.) Pesaro, Terenzi, 1911. 8vo. 10 pp.

Giudice (M. del). Lezioni di aritmetica razionale e algebra elementare. Volume I: Fondamenti della dottrina del numero naturale e frazionario. Roma, Marcolli, 1911. 8vo. 23+184 pp. L. 3.50

Grassi (F.). See Serret (G. A.).

Hannygton (J. C.). Table of logarithms and anti-logarithms. Four figures. London, Layton, 1912. 8vo. 46 pp. Limp. 1s. 6d.

Jones (S. I.). Mathematical wrinkles. A book of knotty problems, mathematical recreations, rules of mensuration, etc. Gunter, Texas, 1912. 12mo. Half leather. $1.50

Koschemann und Otten. Lehr- und Uebungsbuch für den mathematischen Unterricht an Mittelschulen. 2ter Teil: Arithmetik und Algebra. 3ter Teil: Geometrie. Frankfurt, Diesterweg, 1911. M. 4.10

Kurras (K.). Maturitäts-Aufgaben aus der Mathematik nebst ausführlichen Losungen. Berlin, Ulrich, 1912. 8vo. 72 pp. M. 2.00

Lörcher (O.). Methodisches Lehr- und Uebungsbuch der Algebra. Stuttgart, Grub, 1912. 8vo. 7+142 pp. Cloth. M. 2.00

Marks (C. I.). See Mathematical questions.

Mathematical questions and solutions from the Educational Times. Edited by C. I. Marks. New series, volume 20. London, Hodgson, 1912. 8vo. 6s. 6d.

Nardi (P.). Geometria. Parte II: Stereometria. 2a edizione, riveduta. Livorno, Giusti, 1912. 16mo. 72 pp. L. 0.70

Nathing (A.). Lehrbuch der Stereometrie. 2te, unveränderte Auflage. St. Petersburg, Eggers, 1912. 8vo. 3+91 pp. M. 3.00

——. See Belikow (A.).

Pincherle (S.). Lezioni di algebra elementare. Bologna, Zanichelli, 1911. 8vo. 389 pp. L. 4.50

Riboni (G.). Elementi di geometria. 8a edizione, con modificazioni ed aggiunte. Bologna, Zanichelli, 1911. 16mo. 7+366 pp. L. 2.50

Schmehl (C.). Lehrbuch der Stereometrie für höhere Lehranstalten. Giessen, Roth, 1912. M. 2.00

Schmitt (P.). Die Behandlung der Raumlehre in der Volksschule. 3te Auflage. Langensalza, Schulbuchhandlung, 1911. 8vo. 4+151 pp. M. 1.75

Schultze (A.). The teaching of mathematics. New York, Macmillan, 1912. 12mo. Cloth.

Scott (E. E.). Tables of logarithms and anti-logarithms to five places. London, Layton, 1912. 8vo. 383 pp. 5s.

Serret (G. A.). Elementi di trigonometria. Per cura di G. Tolomei. 12a edizione. Firenze, Le Monnier, 1911. 8vo. 150 pp. L. 1.75

——. Trattato di trigonometria. Traduzione, con modificazioni ed aggiunte, da G. Tolomei. Firenze, Le Monnier, 1911. 8vo. 269 pp. L. 2.25

——. Trattato di trigonometria piana e sferica, tradotto da F. Grassi. 7a edizione. Torino, Gallizio, 1911. 8vo. 8+312 pp. L. 3.00

Smith (D. E.). See Suzzallo (H.).

Socci (A.) e Tolomei (G.). Elementi di geometria, secondo il metodo di Euclide. Volume I. 18a edizione. Firenze, Le Monnier, 1912. 8vo. 21+160 pp. L. 1.75

——. Elementi di matematica. Volume I. 6a edizione, completamente rifatta. Firenze, Le Monnier, 1912. 8vo. 4+204 pp. L. 1.50

——. Elementi di matematica. Volume III. 5a impressione. Firenze, Le Monnier, 1911. 8vo. 206 pp. L. 1.50

——. Nozioni intuitive di geometria elementare. 8a impressione. Firenze, Le Monnier, 1912. 8vo. 112 pp. L. 1.00

Suzzallo (H.). The teaching of primary arithmetic. A critical study of recent tendencies in method. With an introduction by D. E. Smith. (Riverside educational monographs.) Boston, Houghton Mifflin, 1911. 16mo. 10+123 pp. $0.60

Tolomei (G.). See Serret (G. A.) and Socci (A.).

Treutlein (P.). See Wiener (H.).

Wells (W.). Complete trigonometry. Revised edition. Boston, Heath, 1911. 12mo. 6+163+9+23 pp. $1.08

Wiener (H.). und Treutlein (P.). Verzeichnis Mathematischer Modelle für Hochschulen. 2te Ausgabe. Leipzig, Teubner, 1912.

III. APPLIED MATHEMATICS.

Abraham (M.). Theorie der Elektrizität. 2 Bände. 1ter Band: Einführung in die Maxwellsche Theorie der Elektrizität. Mit einem einleitenden Abschnitte von A. Föppl. 4te Auflage. Leipzig, Teubner, 1912. 8vo. 18+410 pp. M. 12.00

Alkindi, Tideus und Pseudo-Euklid. Drei optische Werke. Herausgegeben und erklärt von A. A. Björnbo und S. Vogl. Mit einem Gedächtniswort auf A. A. Björnbo von G. H. Zeuthen, einem Verzeichnis seiner Schriften und seinem Bildnis. (Abhandlungen zur Geschichte der mathematischen Wissenschaften. Heft 26, Nr. 3.) Leipzig, Teubner, 1912. 8vo. 176 pp. M. 10.00

BJÖRNBO (A. A.). See ALKINDI.

BRAUER (E. A.). See EULER (L.).

BROWN (Sir H.). Irrigation. Its principles and practice as a branch of engineering. 2d edition, revised. London, Constable, 1912. 8vo. 318 pp. 16s.

CAPPILLERI (A.). See KLAUSER.

COHN (E.). Physikalisches über Raum und Zeit. (Vortrag.) Leipzig, Teubner, 1911. 8vo. 24 pp. M. 0.60

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Sec No
Book No

THE APRIL MEETING OF THE CHICAGO SECTION.

THE thirtieth regular meeting of the Chicago Section of the American Mathematical Society was held at the University of Chicago on Friday and Saturday, April 5–6, 1912, extending through four half-day sessions. The total attendance was seventy-five, including the following fifty-six members of the Society:

Mr. Charles Ammerman, Professor R. P. Baker, Professor G. N. Bauer, Dr. E. G. Bill, Professor G. A. Bliss, Dr. R. L. Börger, Professor W. H. Bussey, Professor R. D. Carmichael, Professor H. E. Cobb, Dr. A. R. Crathorne, Professor D. R. Curtiss, Professor S. C. Davisson, Mr. W. W. Denton, Dr. Arnold Dresden, Professor L. E. Dickson, Professor W. B. Ford, Dr. T. H. Gronwall, Professor A. G. Hall, Professor E. R. Hedrick, Dr. T. H. Hildebrandt, Professor O. D. Kellogg, Professor A. M. Kenyon, Professor Kurt Laves, Professor A. C. Lunn, Dr. G. F. McEwen, Professor Malcolm McNeill, Dr. W. D. MacMillan, Professor G. A. Miller, Professor E. H. Moore, Professor F. M. Morrison, Professor F. R. Moulton, Mr. E. J. Moulton, Professor Alexander Pell, Professor W. D. Pemberton, Professor H. L. Rietz, Professor W. J. Risley, Professor W. H. Roever, Miss Ida M. Schottenfels, Mr. A. R. Schweitzer, Mr. T. M. Simpson, Professor C. H. Sisam, Professor E. B. Skinner, Professor H. E. Slaught, Mr. V. M. Spunar, Dr. E. B. Stouffer, Professor E. J. Townsend, Professor A. L. Underhill, Professor E. B. Van Vleck, Dr. G. E. Wahlin, Mr. C. W. Wester, Professor J. K. Whittemore, Professor E. J. Wilczynski, Professor R. E. Wilson, Professor A. E. Young, Professor J. W. A. Young, Professor Alexander Ziwet.

Professor D. R. Curtiss, chairman of the Section, presided throughout the four sessions, except during the presentation of his own paper, when Professor E. B. Van Vleck occupied the chair.

On Friday evening about fifty members dined together in the café of the University Commons and enjoyed another of those seasons of social intercourse to which all look forward with increasing interest.

At the business meeting on Saturday morning attention was called to the fact that the next meeting of the American Association for the Advancement of Science is to be at Cleveland, Ohio, in December, 1912, and that this occasion will afford an excellent opportunity for holding a general meeting of the American Mathematical Society at the same time and place. In this connection the following resolution was unanimously adopted: That the members of the Society in attendance at the April meeting of the Chicago Section respectfully request the Council to consider the advisability of holding the next annual meeting of the Society in Cleveland in affiliation with the American Association.

Under the head of "informal notes and queries," which was introduced into the programme in order to give the members an opportunity for the interchange of mathematical suggestions not related directly to any of the papers, there was a variety of interesting discussions participated in by Professors Moore, Hedrick, Curtiss, Dickson, Baker, Bliss, Moulton, Lunn, Dr. Bill, Dr. Gronwall, and Miss Schottenfels.

The following papers were presented at this meeting:

(1) Dr. S. LEFSCHETZ: "On birational transformations of three-space related to four-space varieties."

(2) Professor W. H. ROEVER: "Optical interpretations in higher geodesy."

(3) Professor W. H. ROEVER: "Mechanisms for illustrating lines of force."

(4) Professor W. H. ROEVER: "Deviations of falling bodies for a distribution not of revolution. Second paper."

(5) Professor ARNOLD EMCH: "The geometry of conformal rational transformations in a plane."

(6) Professor KURT LAVES: "The present state of the theory of Jupiter's five minor satellites."

(7) Professor D. R. CURTISS: "An extension of Descartes' rule of signs. Second paper."

(8) Professor J. K. WHITTEMORE: "Equality in geometry."

(9) Professor G. O. JAMES: "On the relation between the empirical and the inertial trihedrons of gravitational astronomy."

(10) Professor E. J. WILCZYNSKI: "A forgotten theorem of Newton on planetary motion and an instrumental solution of Kepler's equation."

(11) Dr. G. E. WAHLIN: "The relative number field $K(\sqrt[l]{a})$."

(12) Dr. E. G. BILL: "Analytic curves in non-euclidean space. Second paper."

(13) Professor L. C. Karpinski: "Algebra in the Quadripartitum numerorum of Johannes de Muris."

(14) Professor G. A. Miller: "Infinite systems of indivisible groups."

(15) Mr. W. W. Denton: "Projective differential geometry of developable surfaces."

(16) Professor R. P. Baker: "The method of monodromy and its application to three-parameter quartics."

(17) Professor R. D. Carmichael: "On transcendentally transcendental functions."

(18) Professor R. D. Carmichael: "On the theory of linear difference equations."

(19) Professor J. W. Glover: "A general formula for the valuation of securities."

(20) Dr. E. L. Dodd: "On ordinary plane and skew curves."

(21) Professor Florian Cajori: "On the Spanish symbol *U* for thousands."

(22) Mr. E. W. Chittenden: "Infinite developments and the composition property $(K_{12}B_1)_*$ in general analysis."

(23) Mr. V. M. Spunar: "Note on Mersenne's numbers."

(24) Dr. E. B. Stouffer: "The projective differential geometry of three-spreads generated by ∞^1 planes in five-dimensional space."

(25) Professor E. H. Moore: "Multiplicative interrelations of certain classes of sequences of positive terms."

(26) Dr. T. H. Gronwall: "On a theorem of Fejér's and an analogon to Gibbs's phenomenon."

(27) Dr. T. H. Gronwall: "Some asymptotic expressions in the theory of numbers."

(28) Professor F. R. Moulton: "Closed orbits of ejection and related periodic orbits in the problem of three bodies."

(29) Dr. T. H. Hildebrandt: "Necessary and sufficient conditions for the interchange of limit and summation for a special type of series."

Mr. Chittenden's paper was communicated to the Society through Professor Moore. In the absence of the authors, the papers of Dr. Lefschetz, Professor Emch, Professor James, Professor Karpinski, Professor Glover, Dr. Dodd, Professor Cajori, and Mr. Spunar, were read by title. Abstracts of the papers, except that of Mr. Spunar, follow below in the order of the titles above.

1. It is known that from two depictions of a rational surface we can obtain a birational transformation between two planes, which however is only a product of several quadratic ones. The same holds true for rational four-space varieties, except that the transformations obtained are not in general the product of simpler ones. In this paper Dr. Lefschetz considers the transformations deduced from depictions of quadratic and cubic varieties. The V_3^2 give the well known quadri-quadric transformations, having for base a conic and a point in each space. Several resulting transformations are considered, the simplest being related to a V_3^3 having two conic nodes. This is obtained by projecting the first space on the variety from one of the nodes, and then reprojecting from the other on the second space; it is a cubo-cubic transformation, the fundamental elements being in both spaces a sextic intersection of a cubic and a quadric surface and a point on it. This transformation is then applied to the depiction of a sextic surface having the fundamental sextic for nodal curve and the fundamental point for quadruple point with an elliptic quartic tangent cone, which is obtained from a quadric, and has, as can also be easily verified, $p_g = p_n = P = 0$, which are the conditions of rationality of a surface as given by Castelnuovo.

2. At a point P_0 of the rotating earth, the vertical is the straight line which coincides with a plumb-line of which the bob is at P_0; the meridian plane is the plane which passes through the vertical and is parallel to the axis of rotation of the earth; the east-and-west line is the normal to the meridian plane; the latitude is the complement ϕ of the angle which the vertical (to zenith) makes with the axis (to north); the longitude is the angle λ which the meridian plane makes with a fixed plane through the axis. See Pizzetti, Geodesia Teoretica, 1905, § 5.

On a level surface, that is, a surface at each point of which the normal is the vertical, a curve of constant latitude is a locus for which ϕ is constant, for the equator $\phi = 0$; a curve of constant longitude is a locus for which λ is constant; an east-and-west curve is a locus at each point of which the tangent is the east-and-west line. In general, a tangent to a curve of constant latitude is not an east-and-west line. With the foregoing understanding of the terms involved, Professor Roever shows:

I. The curves of constant latitude are the lines of constant intensity of illumination on a level surface, with respect to a source of light infinitely far away in the direction of the axis. In particular, the equator is the line of shade.

II. The curves of constant longitude are the loci of the brilliant points of the east-and-west curves on a level surface, with respect to infinitely distant pairs of points. That is, if a level surface of the earth, supposed metallic, were scratched along its east-and-west curves, an observer on the moon, or any other distant body in any general direction, would see as a curve of light some curve of constant longitude, the curve being the assemblage of the images of the sun, which may be in any general direction, in the reflecting scratches.

Definitions of brilliant points, and photographs of the curve of light formed by the images of a source of light in the scratches which a circular saw receives during the process of polishing, are given in a paper by the author in the *Transactions*, volume 9, number 3, pages 245–279. A brass sphere, properly scratched, was exhibited in order to show the truth of the second theorem for a special case.

3. In Professor Roever's second paper the first mechanism consists of a pair of overlapping wheels, about 8½ inches in diameter, with radial spokes. These are mounted on parallel axles and rotate in planes which are as near together as possible. These wheels can be made to rotate in the same direction or in opposite directions with equal or different angular velocities. When the wheels rotate rapidly, an observer properly situated will see a well-defined system of curves.

The second mechanism is so designed that the effect is the same as that which would be produced by the first mechanism in the limiting case in which the radius of one of the wheels becomes infinite. The author has already shown for what fields of force the systems of curves exhibited by these mechanisms are the lines of force. See *Transactions of the Academy of Science of St. Louis*, volume 7, numbers 9 and 12, and BULLETIN, volume 12, page 425.

In his Researches in Chemistry and Physics, page 292, Faraday describes the phenomenon observed in viewing one carriage wheel obliquely through another, and also that of viewing a rapidly running carriage wheel through a palisade or railing. Beyond stating that the curves in the first case resemble the lines described by iron filings between the poles of a magnet, Faraday does not identify the curves with any particular fields of force.

4. In a recent paper* Professor Roever derived an expression for the southerly deviation of falling bodies, under the assumption of a distribution of revolution, that is, that the potential function of the earth's gravitational field of force is of the form $f(r, z)$, where r is distance from the axis of rotation and z is that from a fixed plane perpendicular to the axis. He showed that for a potential function for which the standard spheroid is a level surface formula (3) below gives values nearly five times as great as the formula of Gauss,

$$2\omega^2 \sin 2\phi \frac{h^2}{6g}.$$

In the present paper he derives, under the assumption of a distribution not of revolution, the following expressions for the easterly deviation (E. D.) and the southerly deviation (S. D.):

$$(1) \quad \text{E. D.} = \frac{2}{3}\sqrt{2}\omega \cos\phi \frac{h^{\frac{3}{2}}}{\sqrt{W_z}} - \frac{5}{6}\frac{W_{yz}}{W_z}h^2,$$

$$(2) \quad \text{S. D.} = [2\omega^2 \sin 2\phi + 5W_{xz}]\frac{h^2}{6W_z} - \frac{\sqrt{2}}{15}\omega[9\sin\phi W_{yz} + \cos\phi W_{xy}]\frac{h^{\frac{5}{2}}}{W_z^{\frac{3}{2}}},$$

where ω is the angular velocity of the earth's rotation, h is the height through which the body falls, ϕ is the latitude of the initial point P_0, W is the potential function of the relative statical field, that is, the field of force in which the plumb line is in equilibrium, x, y, z are distances measured to the north, east, and downward from P_0, and W_z, W_{xz}, W_{yz}, W_{xy} are the values of the derivatives of W, with respect to the indicated subscripts, at P_0. Hence

$$W_z = g, \quad W_{xz} = \frac{\partial g}{\partial x}, \quad W_{yz} = \frac{\partial g}{\partial y},$$

where g is the acceleration.

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* *Transactions*, vol. 12, No. 3, pp. 335–353.

For a distribution of revolution $W_{yz} = 0$, $W_{xy} = 0$; hence the second terms disappear, being negligible in comparison with the first, that is,

$$\text{S. D.} = \left(2\omega^2 \sin 2\phi + 5\frac{\partial g}{\partial x}\right)\frac{h^3}{6g}.$$

The second derivatives of W which appear in (1) and (2) can be measured experimentally by a method due to Baron Eötvös.* In a certain survey made by Baron Eötvös† values of $\partial g/\partial x$ were found more than ten times as great as the value which corresponds to the potential function which yields the standard spheroid. It is thus seen that it is possible to account for southerly deviations which are nearly fifty times as great as those given by the formula of Gauss.

5. A conformal rational algebraic transformation between two planes may be established by an irreducible rational fraction $w = f(z)/g(z)$ of the complex variable z. Writing $f(z) = u + iv$, $g(z) = r + is$, and separating real from imaginary parts,

$$w = \frac{ru + sv}{r^2 + s^2} + i\frac{rv - su}{r^2 + s^2}.$$

Professor Emch's paper deals with the properties of the system of curves $u = 0$, $v = 0$, $r = 0$, $s = 0$, $ru + sv = 0$, $rv - su = 0$, $u^2 + v^2 = 0$, $r^2 + s^2 = 0$ and their relation to the complex domain. In conclusion some geometrical applications of particular cases are given.

6. This paper forms part of an account of the theory of the satellites of the solar system prepared by Professor Laves for the sixth volume of the Encyklopädie der mathematischen Wissenschaften. It calls attention to the very important recent improvements in our knowledge of this much neglected subject. Indications are not lacking that the theory of Laplace-Souillart must in the near future be replaced by a completely new theory, which abandons the method of variation of constants and takes the equator of Jupiter as the

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* *Encyklopadie der mathematischen Wissenschaften*, Band VI, 1 B., Heft 2, §23, p. 166.

† Verhandlungen der 15. allgemeinen Conferenz der internationalen Erdmessung 1906, 1, p. 337.

fundamental plane, and not Jupiter's orbital plane, which has so far been the adopted plane of reference. It stands to reason that Professor Griffin's paper in volume 9 of the *Transactions* will form the foundation for this new theory for the satellites I, II and III.

7. In a paper read at the April, 1910, meeting of the Section, Professor Curtiss proved the theorem that for any polynomial $f(x)$ there exist polynomials $\phi(x)$ such that the number of variations of sign in the product $f(x)\phi(x)$ is exactly equal to the number of positive roots of $f(x)$. Before this paper was published, an article by Meissner appeared in the *Mathematische Annalen,* discussing the case where $f(x)$ has no positive roots, and introducing certain geometrical methods. This suggested new phases of the problem which are discussed in the present communication. A second proof of the fundamental theorem is given, based on a well-known theorem due to Laguerre. The determination of all cartesian multipliers, $\phi(x)$, of given degree r, is made to depend upon properties of a configuration of linear $(r - 1)$-spaces in space of r dimensions. Preliminary evaluations for the least degree possible for a cartesian multiplier are made, and various formulas are compared in a simple case.

8. The proofs generally given of several theorems of fundamental importance in solid geometry are based on intuitive relations assumed to exist between geometrical and physical objects, and are not admissible in a rigorous treatment of geometry. Among these are: dihedrals having equal plane angles are equal; symmetrical trihedrals are equal if isosceles, otherwise not; two prisms are equal if a trihedral of one is equal to a trihedral of the second, and if corresponding face angles of the equal trihedrals are corresponding angles of equal faces of the prisms.

Professor Whittemore shows that rigorous proofs of these and other theorems may be given on the basis of two simple assumptions and the following definition of equality of two geometrical configurations: Two configurations C and C' are equal if there is a one to one correspondence of all points A of space to all points A' of space such that (1) if A is on C, A' is on C', and conversely, if A' is on C', A is on C; (2) all relations of number and order of points A are identical with the relations of number and order of points A'.

9. In this paper Professor James, following the method of Seeliger, considers the effect of an assumed motion of the empirical trihedron of gravitational astronomy relative to the inertial on the outstanding differences between observation and theory in the motion of the four inner planets as derived by Newcomb. After correction for this motion, the effect of replacing the newtonian law of attraction by one consistent with the relativity theory is studied and the equations of motion of an infinitesimal planet about the sun

$$\frac{d^2x_i}{dt^2} = \frac{k^2M}{r^3}(-x_i + r\epsilon\xi + x_i\phi^2)\left(\frac{1}{\sqrt{1-\phi^2}}\right)^n. \quad (i = 1, 2, 3),$$

set up by Professor de Sitter, are integrated and the exponent n so determined as to remove the excess in the motion of Mercury's perihelion. It is shown that the empirical terms in the motion of the four inner planets then become less than their mean errors.

10. If the mean anomaly M, which is proportional to the time, and the radius vector r of the planet be considered as rectangular coordinates, the (M, r) curve is a trochoid. This theorem, due to Newton, seems to have been almost universally forgotten. Professor Wilczynski points out that this theorem is important in the first place as giving an intuitive notion of the functional relation between the radius vector and the time, and in the second place as offering the possibility for the construction of a simple instrument by means of which r may be obtained without any calculation whatever. Thus, the solution of Kepler's equation would become altogether unnecessary if the instrument were constructed with the requisite degree of accuracy. In all cases, however, even a moderately accurate machine of this kind would be of great assistance by giving a close approximation to the value of the eccentric anomaly E, the final value then to be obtained by the application of the usual differential formulas.

11. If we denote by k any algebraic number field, and let α be any integer in k which is not the lth power of an integer in the same field, where l is a rational odd prime, then $\sqrt[l]{\alpha}$ determines a field K of degree l relative to k. Dr. Wahlin gives in this paper a complete determination of the decom-

position of the prime ideals of k into their prime factors in K. The paper is a generalization of the one presented by the author at the April meeting, 1911, as well as of the first part of the article by Westlund in the *Transactions*, 1910. The same problem for the case $l = 2$ is solved in an article by Hilbert in the *Mathematische Annalen*, volume 51.

12. This paper is a continuation of the work which Dr. Bill presented at the December, 1911, meeting of the Chicago Section. Having classified all the analytic curves of non-euclidean space and derived the Frenet-Serret differential equations for the so-called regular curves, he comes to a discussion of the natural equations of this latter set of curves.

13. Johannes de Muris, who flourished in the first half of the fourteenth century, wrote extensively on music as well as mathematics. His "Quadripartitum numerorum" has been referred to by many writers, but only a small portion of the second book dealing with arithmetic has been published. Professor Karpinski continues his studies in medieval algebra by a discussion of the contents of the third book of the Quadripartitum, which deals with algebra. This book is entitled "Novus algorismus de additis et diminutis." The study shows that de Muris drew not only from Al-Khowarizmi, but further that he took many problems almost word for word from Leonardo of Pisa. Inserted between the third and the fourth books is a so-called Semi liber which treats of the extraction of roots and again follows Leonardo. This study is based upon the Codex Pal. Vind. (Vienna), 4770, in which the third book is found on pages 229ver.—261rec.

14. A group is said to be divisible if it is the direct product of two or more groups none of which is identity. If it is not such a direct product it is said to be indivisible or prime. If a group is divisible it can be resolved into indivisible factor groups in essentially only one way. Professor Miller considers mainly various infinite systems of indivisible groups which are such that no two groups of the system have the same order. Among the theorems established are the following: A necessary and sufficient condition that a solvable group is a direct product of a Sylow subgroup and another subgroup is that its group of inner isomorphisms involves the corresponding

Sylow subgroup as a factor of a direct product, whenever it involves such a Sylow subgroup. A group of order p^m, p being any prime number, whose central is cyclic is always indivisible. A necessary and sufficient condition that the holomorph of a cyclic group is indivisible is that the order of this cyclic group is a power of a single prime number.

15. In this paper Mr. Denton presents the details of a projective differential theory of developables, which was indicated by Professor Wilczynski at the April, 1909, meeting of the Chicago Section.*

16. Professor Baker's paper has for its aim the establishment of an effective technique. It is proved that ordinary points on the discriminantal locus give rise to simple transpositions as elements of the monodromy group. The singularities, including the intersection of the various sheets in the case of reduction, give any other types of substitution which arise. By birational transformation of the resolvent equations three parameter quartics of each of the transitive groups are made available as material. In the case of non-symmetric groups, this can be so arranged that one parameter is removable by a rational process and is ineffective for monodromy. Examples are given of all cases of irreducible three-parameter quartics with assigned group and assigned invariant subgroup as monodromy group; also of the determination of the algebraic number whose adjunction reduces the group to the monodromy group.

17. Following the classification made by Moore in his fundamental paper † on transcendentally transcendental functions Professor Carmichael calls a transcendental function y of x algebraically or transcendentally transcendental according as it does or does not satisfy an algebraic differential equation. In the first part of the paper attention is confined to functions y of x expansible in the form

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots,$$

where $a_0, a_1, a_2, \cdots$ are rational fractions in their lowest terms.

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* BULLETIN, vol. 15, p. 436.

† *Math. Ann.*, vol. 48, pp. 49–74.

Among the general results obtained the following may be mentioned: (1) If y is an integral of an algebraic differential equation, then there exists an integer τ such that, for increasing n, the denominator of a_n ultimately becomes and remains less than n^{τ^n}. (2) If y is an integral of an algebraic differential equation and if the denominator of a_n is written in the form $r_n s_n$, where s_n is prime to $n!$ and to the denominator of every a_i, $i < n$, then there exists an integer ϵ such that, for increasing n, s_n ultimately becomes and remains less than $n\epsilon$. (3) Let $b_0, b_1, b_2, \cdots,$ be a set of rational fractions in their lowest terms, an infinite number of which are different from zero; let them have the property that a number N exists such that, for $n > N$, neither the numerator nor the denominator of b_n contains a prime factor greater than n. Let α and β be relatively prime positive integers. Then the series

$$y = b_0 + \frac{b_1}{(\alpha - \beta)^k} x + \cdots$$
$$+ \frac{b_n}{(\alpha - \beta)^{k_{1n}}(\alpha^2 - \beta^2)^{k_{2n}} \cdots (\alpha^n - \beta^n)^{k_{nn}}} x^n + \cdots$$

if it converges, defines a function which is transcendentally transcendental. Here k_{ij} is an integer and $k_{nn} \geqq 1$ for an infinite number of values of n for which $b_n \neq 0$. (4) Particular examples of functions coming under this theorem are those which are analytic at zero and which satisfy the following functional equations, where q is rational and $|q| \neq 1, 0$:

$$g(qx) = (1+x)g(x), \quad g(qx) - g(x) = \frac{x}{1-x}, \quad e^x - 1, \quad \sin x, \quad \text{etc.}$$

In the second part of the paper it is shown that every analytic function integral of the equation

$$g(qx) = (1 + x)g(x), \quad |q| \neq 1, 0,$$

is transcendentally transcendental. Finally the following important result is obtained: Two essentially different classes of transcendentally transcendental functions are defined by linear difference equations with rational coefficients and by linear q-difference equations with rational coefficients.

18. Guichard, Appell, and Hurwitz have each proved by indirect means the important theorem that if $G(x)$ is any

entire function then there exists another entire function $u(x)$ verifying the relation

$$u(x+1) - u(x) = G(x).$$

In the first part of his second paper Professor Carmichael proves this theorem by direct means; that is to say, a power series is assumed for $u(x)$ and substitution is made in the above equation. There results an infinite system of equations for determining the coefficients of this power series. This system is solved by what is believed to be a novel method of importance, and in such way that the resulting function $u(x)$ is entire.

The remainder of the paper is devoted to certain generalizations of the above result.

19. Professor Glover derives a formula for the valuation of a very general type of security. The security is redeemed in r equal instalments at intervals of t years, the first redemption being made after f years. The annual rate of dividend is g payable in m instalments and the security is purchased to realize the investor a nominal rate of interest with frequency of conversion m. The value is expressed in terms of a single function, namely, the present values of immediate annuities certain.

20. Dr. Dodd shows that the conditions of continuity, differentiability, and convexity imposed by Pringsheim[*] in defining an ordinary arc and its function $y=f(x)$ are not coextensive with the conditions used by Du Bois-Reymond,[†] who, in place of convexity, requires that there should not be an infinite number of maxima with respect to any straight line. But Pringsheim's conditions necessitate the continuity and univariance of the first derivative except at the end points of the interval, where being univariant it may become infinite. Pringsheim's ordinary arc is Osgood's[‡] regular arc defined by

$$x = \varphi(t), \quad y = \psi(t), \qquad (t_0 \leqq t \leqq t'),$$

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[*] Encyklopädie, II, A. 1, p. 22.

[†] *Crelle*, vol. 79, p. 32.

[‡] Lehrbuch der Funktionentheorie, p. 43.

where φ and ψ are continuous and have continuous first derivatives which do not simultaneously vanish in (t_0, t'), with the added condition that ψ'/φ' be univariant in (t_0, t').

The definition of an ordinary arc is then restated to make it invariant to rotation, a generalization for three dimensions is given, and finally straight lines, ordinary plane arcs, and ordinary skew arcs are distinguished by means of the Wronskians of the first derivatives of the defining functions, and the minors of the Wronskians.

21. Professor Cajori points out that, at the beginning of the nineteenth century and for three hundred years previous, Spanish-American manuscripts often contain, along with the Hindu-Arabic numerals, symbols taking the form of a U or V or O, or sometimes an old form of the Greek letter sigma, to designate "thousands." Thus, 7U291 = 7291.

22. In the "Introduction to general analysis" E. H. Moore has formulated a theory of developments of a general range P. A development of P is defined to be a denumerably infinite sequence of stages, each stage consisting of a finite system of subclasses of P. In terms of a development Moore and A. D. Pitcher have secured a body of theorems relating to classes of functions on a general range. Mr. Chittenden considers developments whose stages are denumerably infinite systems of subclasses and obtains theorems which include those of Moore and Pitcher and cover additional cases.

24. The projective differential geometry of ruled surfaces in ordinary space has been developed by Professor Wilczynski by means of a system of two linear homogeneous differential equations of the second order. In the present paper Dr. Stouffer develops by similar methods the projective differential geometry of three-spreads generated by ∞^1 planes in five-dimensional space by making use of a system of three linear homogeneous differential equations of the second order. The three-spreads for which any two consecutive generating planes intersect are excluded from the theory.

25. Consider sequences α: $a_1, a_2, a_3 \cdots$ of positive terms and denote by $\mathfrak{C}_u$, where u is a non-vanishing real number, the class of all sequences α whose u power sequences α^u of positive terms

have convergent sums $\mathfrak{a}_1^u + \mathfrak{a}_2^u + \mathfrak{a}_3^u + \cdots$, and by $\mathfrak{D}_u$ the class of all sequences α whose u power sequences α^u have divergent sums. Then, d, e, f being three positive numbers such that $(d + e)f = d + e$, we have the multiplication theorems

$$\mathfrak{C}_d\mathfrak{C}_e = \mathfrak{C}_f, \quad \mathfrak{C}_e\mathfrak{D}_{-f} = \mathfrak{D}_{-d}.$$

These theorems are convenient normalizations, slightly generalized, of theorems due to Pringsheim* and Landau,† which especially for $(d, e, f) = (2, 2, 1)$ play a rôle in the theory of functions of infinitely many variables.

Professor Moore determines a closed multiplication table for a set of classes of sequences of positive terms whose definitions involve, besides the notions u power and convergence, the notions finiteness and exponent of convergence.

26. Let $f(x)$ be continuous and $|f(x)| \leqq M$ for $0 \leqq x \geqq 2\pi$, and suppose that the Fourier coefficients of $f(x)$ satisfy the inequalities

$$|a_n| \leqq A/n, \quad |b_n| \leqq B/n \quad (n = 1, 2, \cdots).$$

Let

$$s_n(x) = a_0 + a_1 \cos x + b_1 \sin x + \cdots + a_n \cos nx + b_n \sin nx$$

be a partial sum of the Fourier series for $f(x)$; Fejér has shown that

$$|s_n(x)| \leqq M + \frac{1}{n+1}\left(A\sum_{v=1}^{n}|\cos vx| + B\sum_{v=1}^{n}|\sin vx|\right),$$

whence

$$|s_n(x)| < M + A + B.$$

In Dr. Gronwall's paper, this inequality is replaced by the following closer ones:

$$|s_n(x)| < M + \sqrt{A^2 + B^2}$$

for any values of A and B, and

$$|s_n(x)| < M + \tfrac{1}{2}A + \frac{\sqrt{3}}{2}B$$

for $B \geqq \sqrt{3}A$.

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* *Münchener Berichte*, vol. 32 (1902), p. 177.

† *Göttinger Nachrichten*, 1907, pp. 25–27.

Making

$$U_n(x) = \frac{1}{n+1}(|\cos x| + |\cos 2x| + \cdots + |\cos nx|),$$

$$V_n(x) = \frac{1}{n+1}(|\sin x| + |\sin 2x| + \cdots + |\sin nx|),$$

it is further shown that $\lim\limits_{n=\infty} U_n(x)$ and $\lim\limits_{n=\infty} V_n(x)$ are pointwise discontinuous functions of x, having the following values:
For $x = 0$,

$$\lim_{n=\infty} U_n(x) = 1, \quad \lim_{n=\infty} V_n(x) = 0;$$

for $x = \frac{p}{q}\pi$, where p ($\neq 0$) and q are integers and relative primes,

$$\lim_{n=\infty} U_n(x) = \frac{1}{2q}\left(\cot\frac{\pi}{4q} + (-1)^{q+1}\tan\frac{\pi}{4q}\right),$$

$$\lim_{n=\infty} V_n(x) = \frac{1}{q}\cot\frac{\pi}{2q};$$

for $x : \pi$ irrational,

$$\lim_{n=\infty} U_n(x) = \lim_{n=\infty} V_n(x) = \frac{2}{\pi}.$$

It is finally shown that

$$\tfrac{1}{2} = \text{min.} \lim_{n=\infty} U_n(x) = \lim_{n=\infty} \text{min.}\, U_n(x) < \text{max.} \lim_{n=\infty} U_n(x) =$$

$$= \lim_{n=\infty} \text{max.}\, U_n(x) = 1,$$

$$0 = \text{min.} \lim_{n=\infty} V_n(x) = \lim_{n=\infty} \text{min.}\, V_n(x) < \text{max.} \lim_{n=\infty} V_n(x) =$$

$$= \frac{2}{\pi} < \lim_{n=\infty} \text{max.}\, V_n(x) = 0.724 \cdots,$$

$$\text{max.}\ V_n(x) < \text{max.}\ V_{n+1}(x),$$

so that $V_n(x)$ presents an analogue to Gibbs' phenomenon, while there is none in the case of $U_n(x)$.

27. Let

$$s_\alpha(x) = \sum_{d/x} d^\alpha$$

be the sum of the αth powers of all divisors of the integer x. Dr. Gronwall shows that

$$\limsup_{x=\infty} \frac{s_\alpha(x)}{x^\alpha} = \xi(\alpha) \quad \text{for } \alpha > 1,$$

$$\limsup_{x=\infty} \frac{s_\alpha(x)}{x \log \log x} = e^C \quad \text{for } \alpha = 1,$$

$$\limsup_{x=\infty} \frac{\log s_\alpha(x)/x^\alpha}{(\log x)^{1-\alpha}/\log \log x} = \frac{1}{1-\alpha} \quad \text{for } 0 < \alpha < 1,$$

where $\xi(\alpha)$ is the Riemann zeta function and C the Eulerian constant.

28. In this paper Professor Moulton proves the existence of orbits of ejection which are also orbits of collision with the same finite body. These orbits are the limits of two classes of periodic orbits.

In his Stockholm lectures Painlevé conjectured that an orbit satisfying a certain analytic condition would be an orbit of ejection or collision. In this paper it is proved that any point and an arbitrary positive velocity may be selected, and that an orbit of collision and one of ejection exist which pass through the point with the given velocity. The direction of motion is an analytic function of the coordinates of the point and the velocity.

29. The chief theorem in Dr. Hildebrandt's paper is the following: Suppose $m > 0$,

$$\sum_p | x_{np} |^m$$

convergent for every value of n, and

$$\lim_n x_{np} = x_p$$

for every value of p. Then the necessary and sufficient condition that

$$\sum_p | x_p |^m$$

be convergent and

$$\lim_n \sum_p | x_{np} |^m = \sum_p | x_p |^m$$

is that

$$\lim_n \sum_p | x_{np} - x_p |^m = 0.$$

In the proof of this theorem there arises another necessary and sufficient condition concerning itself with the limit of the sum of a sequence of series of positive terms.

H. E. SLAUGHT,
Secretary of the Section.

THE TWENTY-FIRST REGULAR MEETING OF THE SAN FRANCISCO SECTION.

THE twenty-first regular meeting of the San Francisco Section of the Society was held at Stanford University, on Saturday, April 6, 1912. About fifteen persons were present, including the following members of the Society:

Professor R. E. Allardice, Mr. B. A. Bernstein, Dr. Thomas Buck, Professor H. F. Blichfeldt, Professor G. C. Edwards, Professor M. W. Haskell, Professor L. M. Hoskins, Professor D. N. Lehmer, Professor H. C. Moreno, Professor E. W. Ponzer.

Morning and afternoon sessions were held, Professor Hoskins, chairman of the Section, presiding.

The following papers were presented at this meeting:

(1) Mr. B. A. BERNSTEIN: "On the relation between spaces in n-dimensional space and their concrete representation for the space of four dimensions."

(2) Dr. THOMAS BUCK: "Some periodic orbits of three finite bodies."

(3) Dr. S. LEFSCHETZ: "On cubic surfaces and their nodes."

(4) Professor H. F. BLICHFELDT: "On the order of linear homogeneous groups. Fifth paper."

(5) Mr. B. A. BERNSTEIN: "On an algebra of probability" (preliminary communication).

In the absence of the author the paper by Dr. Lefschetz was read by title. Abstracts of the papers follow.

1. In this paper Mr. Bernstein derives synthetically theorems concerning the fundamental relations between spaces in a hyperspace, including general formulas relating to the dimensionality of any space in any other space as element. He then shows how those relations can be represented concretely for four-dimensional space.

2. The orbits considered by Dr. Buck are obtained by the method of continuation as to a parameter which has been developed by Professor Moulton in connection with similar problems. The results are given as power series in the parameter which are known to converge for all values of the parameter sufficiently small. When the parameter is put equal to zero the solutions found reduce to the circular Lagrangian equilateral triangle solution of the problem of three bodies. For the other admissible values of the parameter the motion may be described as an oscillation about the vertices of this triangle.

3. In this paper Dr. Lefschetz gives a treatment of the cubic surface S^3 derived from a generation given by R. Sturm. The surface is obtained as the locus of the variable twisted cubic common to two quadrics which pass respectively through two other twisted cubics R_1^3 and R_2^3, having a point O in common, and which also pass through a common variable line through O. It is shown by the principle of correspondence that both curves have 6 common bisecants, not going through O, and which are therefore on S^3. The existence of the other 21 lines on S^3 and their properties are readily established. The modifications necessary in order to obtain the 21 non-ruled cubic surfaces are given except for the surface U_8 (Schlafli's classification—Salmon, Geometry of Three Dimensions, page 395), for which it is here proved that it contains no twisted cubic, being the only S^3 having this property. For the three surfaces with a node (U_6, U_7, U_8) a second construction is given, based on a modification of the construction of an S^3 by projective pencils of quadrics and planes (Steiner). In addition to their being new, the interest in these constructions consists first, in the geometric classification of cubic surfaces, according to the mutual position of R_1^3 and R_2^3 in space, and second, in making it possible to show the properties of each of the surfaces without considering them as limiting cases. This is illustrated for the case of the surface with a binode.

4. On the basis of a principle discovered by Bieberbach, Frobenius proved the following theorem (*Berliner Sitzungsberichte*, 1911, page 373): Let a finite group G of linear homogeneous substitutions contain a substitution S whose "multipliers" (i. e., the roots of its "characteristic equation," necessarily roots of unity) $\alpha_1, \alpha_2, \cdots, \alpha_n$ possess the property arc $(\alpha_i/\alpha_j) < 60°$; then all the substitutions of G of this character are mutually commutative. It follows that G is not primitive. Professor Blichfeldt's paper, based on the same principle, contains a number of theorems of which the following are noted here:

(1) If arc $(\alpha_i/\alpha_j) \leqq 72°$, then G is not primitive.

(2) No abelian subgroups of a primitive collineation group in n variables can be of order $\geqq 5^{n-1}$.

This maximum order, of importance in fixing a limit to the order of collineation groups, is presented in much lower (and correspondingly much more complex) form for large values of n. A more general theorem is given, of which the following is a special case:

(3) If the characteristic equation of a substitution of prime order p has only three distinct roots, then $p \leqq 13$, or the group is not primitive.

5. The theory of probability has underlying it both numerical and logical assumptions. Mr. Bernstein, in his second paper, presents a set of postulates for a numerico-logical theory of probability and establishes on this foundation a general method of solving the problem: Given $p_1, p_2, \cdots, p_n$, the probabilities of the events $E_1, E_2, \cdots, E_n$; to find the probability x of the event $E = f(E_1, E_2, \cdots, E_n)$.

T. M. PUTNAM,
Secretary of the Section.

IMPLICIT FUNCTIONS DEFINED BY EQUATIONS WITH VANISHING JACOBIAN.

BY MR. G. R. CLEMENTS.

(Read before the American Mathematical Society, April 28, 1911.)

LET

$$x_i = f_i(y_1, \cdots, y_n) \qquad (i = 1, 2, \cdots, n)$$

be n functions, each analytic in the neighborhood of the point $(y) = (0)$. The case in which the Jacobian

$$J = \sum \pm \frac{\partial f_1}{\partial y_1} \frac{\partial f_2}{\partial y_2} \cdots \frac{\partial f_n}{\partial y_n}$$

does not vanish in the point $(y) = (0)$, and the case in which J vanishes identically in the neighborhood of this point, are familiar. I wish to state here some theorems which I have obtained for the intermediate case that J vanishes in the point $(y) = (0)$, but does not vanish identically in the n variables y. For a number of the theorems n is restricted to the value 2.

If $x_1, \cdots, x_n$ denote n independent complex variables, any region which constitutes at least the totality of all points $(x_1, x_2, \cdots, x_n)$ for which

$$| x_1 - a_1 | < \epsilon, | x_2 - a_2 | < \epsilon, \cdots, | x_n - a_n | < \epsilon$$

for some $\epsilon > 0$, is called a *complete* neighborhood of the point $(a_1, a_2, \cdots, a_n)$.

Any region which satisfies the conditions that

(1) for every $\epsilon > 0$, it contains points $(x_1, x_2, \cdots, x_n)$ for which

$$| x_1 - a_1 | < \epsilon, | x_2 - a_2 | < \epsilon, \cdots, | x_n - a_n | < \epsilon;$$

(2) for no $\epsilon > 0$, *every* point $(x_1, x_2, \cdots, x_n)$ for which

$$| x_1 - a_1 | < \epsilon, | x_2 - a_2 | < \epsilon, \cdots, | x_n - a_n | < \epsilon$$

is a point of the region

is called a *partial* neighborhood of the point $(a_1, a_2, \cdots, a_n)$.

For purposes of simplicity, the neighborhood of the point $(0, 0, \cdots, 0)$ is considered throughout this paper. In what follows, the unqualified word *neighborhood* is used to mean *complete neighborhood.*

Consider the transformation

$$T: \qquad x = f(u, v), \quad y = \varphi(u, v),$$

(*a*) $f(u, v)$ and $\varphi(u, v)$ denoting functions of the complex variables u and v, single-valued and analytic throughout a neighborhood R of $u = 0, v = 0$, and satisfying the conditions

$$(b) \qquad f(0, 0) = 0, \quad \varphi(0, 0) = 0,$$

$$(c) \qquad J(u, v) \equiv \begin{vmatrix} f_u & f_v \\ \varphi_u & \varphi_v \end{vmatrix} \not\equiv 0, \quad J(0, 0) = 0.$$

To any point (u, v) of R, there corresponds, under T, one and but one point (x, y) of the (x, y) space. The totality of points (x, y) which thus correspond to points (u, v) of R, constitute a finite region $\bar{R}$ of the (x, y) space. In general, more than one point (u, v) will yield the same pair of values for x and y. We shall count a point (x, y) but once, thus regarding it as completely characterized by its coordinates, and seek an inverse transformation

$$u = \bar{f}(x, y), \quad v = \bar{\varphi}(x, y)$$

which will put in evidence all those points (u, v) of R which correspond under T to any point with coordinates (x, y) of $\bar{R}$. Thus the functions $\bar{f}(x, y)$, $\bar{\varphi}(x, y)$ are to be defined only in the points of $\bar{R}$, and there uniquely or as multiple-valued functions.

THEOREM I. *The transformation T can never have an inverse*

$$u = \bar{f}(x, y), \quad v = \bar{\varphi}(x, y)$$

such that $\bar{f}(x, y)$ and $\bar{\varphi}(x, y)$ are both analytic throughout a complete neighborhood of $x = 0, y = 0$.

The theorem holds for any value of n.

THEOREM II. *If in the transformation T, $f(u, v)$ and $\varphi(u, v)$ have no common factor in the point $(0, 0)$, then there exists an inverse, defined throughout a complete neighborhood of $x = 0$, $y = 0$, everywhere continuous, finitely multiple-valued, analytic except along a complex one-dimensional locus, and having the value $u = 0$, $v = 0$, when $x = 0, y = 0$.*

If instead of the transformation T, we consider the transformation T', whose definition differs from that of T in that it omits the condition (c), we deduce

THEOREM III. *If to each point* (x, y) *of the region* $\bar{R}$ *there corresponds from* T' *a single point* (u, v) *of a region* R, *then* u *and* v *are analytic in* x *and* y *in the point* $x = 0, y = 0$.

It follows from Theorem I and from Theorem III that there can not exist a transformation T for which the inverse is single-valued throughout any neighborhood $\bar{R}$ of the point $x = 0, y = 0$.

The proof for Theorem II and for Theorem III, depends on the Weierstrassian implicit function theorem.[*] Theorem IV is a generalization of this.

THEOREM IV. *The system of equations*

$$\begin{aligned} f_1(x_1, \cdots, x_n; y_1, \cdots, y_p) &= 0, \\ f_2(x_1, \cdots, x_n; y_1, \cdots, y_p) &= 0, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad &\cdot\quad\cdot \\ f_p(x_1, \cdots, x_n; y_1, \cdots, y_p) &= 0, \end{aligned}$$

where

(1) f_i *is an analytic function of its arguments throughout a neighborhood* R *of* $(x) = 0, (y) = 0$;

(2) $f_i(0, \cdots, 0; 0, \cdots, 0) = 0 \quad (i = 1, 2, \cdots, p)$;

$$(3) \quad J_1 \equiv \frac{D(f_1, f_2, \cdots, f_p)}{D(y_1, y_2, \cdots, y_p)} = 0 \text{ when } (x) = 0, (y) = 0,$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$J_{k-1} \equiv \frac{D(J_{k-2}, f_2, \cdots, f_p)}{D(y_1, y_2, \cdots, y_p)} = 0 \text{ when } (x) = 0, (y) = 0,$$

$$J_k \equiv \frac{D(J_{k-1}, f_2, \cdots, f_p)}{D(y_1, y_2, \cdots, y_p)} \neq 0 \text{ when } (x) = 0, (y) = 0,$$

defines $y_1, y_2, \cdots, y_p$ *as functions of* $x_1, x_2, \cdots, x_n$, *continuous throughout the neighborhood of* $(x) = 0$. *With a suitable counting of multiplicities everywhere present, this solution is* k*-fold; it is analytic with the possible exception of complex* $(n - 1)$*-dimensional loci where determinations of the root, in general distinct, become coincident. When* $(x) = 0, (y) = 0$.

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[*] Weierstrass, Abhandlungen aus der Funktionenlehre, p. 105; Bliss, BULLETIN, April, 1910, p. 356; Macmillan, ibid., December, 1910, p. 116.

THEOREM V. *The transformation*

$$x_1 = f_1(y_1, \cdots, y_n), \quad x_2 = f_2(y_1, \cdots, y_n), \cdots, x_n = f_n(y_1, \cdots, y_n),$$

where

(1) f_i *is a single-valued and analytic function of* $y_1, \cdots, y_n$ *in the point* $(0, \cdots, 0)$ *and vanishes there* $(i = 1, 2, \cdots, n)$;

$$(2) \qquad J_1(0, \cdots, 0) = \left.\frac{D(f_1, \cdots, f_n)}{D(y_1, \cdots, y_n)}\right|_{(y)=0} = 0,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$J_{k-1}(0, \cdots, 0) = \left.\frac{D(J_{k-2}, f_2, \cdots, f_n)}{D(y_1, y_2, \cdots, y_n)}\right|_{(y)=0} = 0,$$

$$J_k(0, \cdots, 0) = \left.\frac{D(J_{k-1}, f_2, \cdots, f_n)}{D(y_1, y_2, \cdots, y_n)}\right|_{(y)=0} \neq 0,$$

has a k-valued continuous inverse, defined throughout a complete neighborhood of $(x) = 0$. *This inverse is analytic, with k distinct determinations, except along a complex* $(n-1)$*-dimensional locus, where it is continuous and less than k-valued. When* $(x) = 0$, $(y) = 0$.

The remaining theorems have to do with a detailed study of the transformation T. Theorems VI, VII, and VIII yield a complete discussion for the case in which at least one element of the determinant J, say f_u, is not zero at the origin.

THEOREM VI. *If the transformation T satisfies the further conditions*

(1) $f_u(0, 0) \neq 0$,

(2) $f(u, v)$ *not a factor of* $\varphi(u, v)$ *in the point* $u = 0$, $v = 0$, *then, for the sequence of functions*

$$J_1(u, v) \equiv \frac{D(f, \varphi)}{D(u, v)}, \cdots, J_{i+1}(u, v) \equiv \frac{D(f, J_i)}{D(u, v)},$$

there exists an integer $k \geqq 2$ *such that*

$$J_1(0, 0) = \cdots = J_{k-1}(0, 0) = 0, \quad J_k(0, 0) \neq 0,$$

and for the transformation there exists a k-valued continuous inverse, defined throughout a complete neighborhood of $x = 0$, $y = 0$. *This inverse is analytic, with k distinct determinations, except along a complex one-dimensional locus, where it is continuous and less than k-valued. Finally* $u = 0$, $v = 0$ *when* $x = 0$, $y = 0$.

THEOREM VII. *If the transformation T is of the form*

$$x = f(u, v), \quad y = [f(u, v)]^n g(u, v),$$

where

$$f_u(0, 0) \neq 0, \quad g(0, 0) = 0,$$

and where $f(u, v)$ is not a factor of $g(u, v)$ in the point $u = 0$, $v = 0$, then to the point $x = 0, y = 0$ corresponds the locus $f(u, v) = 0$, and to any other point (x, y)

(1) *if $f_u g_v - f_v g_u \neq 0$ when $u = 0$, $v = 0$, there corresponds one and only one point (u, v), the relationship being defined by functions of x and y, single-valued and analytic in this point;*

(2) *if $f_u g_v - f_v g_u = 0$ when $u = 0$, $v = 0$, there correspond in general m points, $m \geqq 2$, the relationship being defined by m-valued continuous functions of x and y.*

THEOREM VIII. *A transformation T for which $f(u, v)$ is a factor of $\varphi(u, v)$, can be replaced by a transformation*

$$u_1 = f(u, v), \quad v_1 = h(u, v),$$

where $h(u, v)$ is a single-valued and analytic function of u and v in the point $(0, 0)$ and vanishes there, and where $f(u, v)$ is not a factor of $h(u, v)$ in this point, followed by a finite number m of transformations of the form

$$x = u, \quad y = uv,$$

combined with not more than m translations of the form

$$x = u, \quad y = v + c.$$

THEOREM IX. *A transformation T which has the form*

$$x = f(u) = a_n u^n + a_{n+1} u^{n+1} + \cdots \quad (a_n \neq 0),$$
$$y = \varphi(u, v),$$

can be replaced by two transformations, $T = b \cdot a$, where

$$\begin{array}{lll} a: & u_1 = u\sqrt[n]{a_n + a_{n+1}u + \cdots}, & v_1 = \varphi(u, v), \\ b: & x = u_1^n, & y = v_1. \end{array}$$

The inverse of (a) is defined by analytic functions, or can be determined from Theorems VI, VII *and* VIII.

THEOREM* X. *If $k = 2$ in Theorem* VI, *the transformation*

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* Cf. L. S. Dederick, Harvard doctoral thesis (1909), "Certain singularities of transformations of two real variables," p. 124.

T can be replaced by three transformations which are one-to-one and analytic both ways, combined with one transformation of the form

$$x = u, \quad y = v^2.$$

THEOREM XI. *If the transformation T has the form*

$$x = f(u, v) \equiv c_{20}u^2 + c_{11}uv + c_{02}v^2 + c_{30}u^3 + \cdots,$$

$$y = \varphi(u, v) \equiv d_{20}u^2 + d_{11}uv + d_{02}v^2 + d_{30}u^3 + \cdots,$$

where the terms quadratic in u and v are not identically zero for either f or φ, and where these quadratic terms have no common factor, then there exists a four-valued continuous inverse, defined throughout the complete neighborhood of $x = 0$, $y = 0$. This inverse is analytic, with four distinct determinations, except along a complex one-dimensional locus, where it is continuous and less than four-valued. Finally, $u = 0$, $v = 0$ when $x = 0$, $y = 0$.

MADISON, WIS.

DARWIN'S SCIENTIFIC PAPERS.

Scientific Papers. By SIR GEORGE HOWARD DARWIN, K.C.B., F.R.S., Plumian Professor in the University of Cambridge. Vol. III, Figures of Equilibrium of Rotating Liquid and Geophysical Investigations, xv + 527 pp.; Vol. IV, Periodic Orbits and Miscellaneous Papers, xviii + 592 pp. Cambridge University Press, 1910, 1911. Royal 8vo.

THE first two volumes of Sir George Darwin's researches have already been reviewed in the columns of the BULLETIN.[*] They contained papers on the practical and theoretical tidal problems which the oceans present and his earlier attacks on the past history of the earth-moon system. The third and fourth volumes contain his investigations on the relations of fluid masses in rotation about an axis under gravitational forces, on the periodic orbits which a particle can describe when attracted by two bodies of finite masses moving in circular orbits about one another, and a number of papers on other matters.

Of the forms which a single mass of fluid can take when revolving without relative motion about an axis under its own gravitational attraction only, two have long been known. Maclaurin's ellipsoid is one of revolution about the axis of rotation and its eccentricity will have a value which depends on the rate of angular rotation. Jacobi's ellipsoids have three unequal axes. These forms are of little interest unless the motion is stable. As the eccentricity of the Maclaurin ellipsoid increases, the series reaches a point where stability changes to instability. At this place the Jacobian series changes from having three unequal axes to an ellipsoid having the two axes in the equatorial plane equal, that is, to the Maclaurin ellipsoid and, what is more important, from stability to instability.

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[*] Vol. 16, pp. 73–78.

If the stable portion of the Jacobian series be now followed from this point, there comes a place where the motion changes from stability to instability and at this place it crosses a third series, known as the pear-shaped figures of equilibrium which there degrade into a Jacobian ellipsoid. These pear-shaped figures were first known through a memoir by M. Poincaré which appeared in 1885.

Is it possible to carry the investigation further along the same lines so as to discover figures of relative equilibrium in which, as a certain constant alters, the fluid mass shall separate into two parts, the motion remaining stable? From the point of view of the development of the earth-moon and similar systems, this is the principal question.

While Poincaré was carrying the development to the stage mentioned above, Darwin had already attacked the problem from the other end. In the paper "On figures of equilibrium of rotating masses of fluid" he considers two equal masses revolving near to one another in relative equilibrium and examines the forms they will assume as they approach towards contact. By the use of spherical harmonic analysis he carries the solution to the stage where they are not only in contact but actually overlap. The last is of course a physical impossibility but it leads directly to the possibility of the existence of a figure formed of two masses joined by a narrow neck. The stability question is not, however, settled at this time. Several cases where the two masses are unequal are also considered.

In the attempt to use spherical harmonic analysis, the numerical applications become doubtful when the deviation from a sphere is too great. Darwin therefore undertakes a laborious and detailed investigation into the properties of ellipsoidal harmonics, so that a start may be made with an ellipsoid instead of with a sphere, and that the deviations from the ellipsoidal form may be computed. More than a quarter of Volume III is devoted to this subject, but the space is not extravagant in view of the fact that the papers which follow mainly owe their success to the completeness with which he has developed the theory of these complicated functions and to the methods he has adopted in order that numerical results might be deduced from them.

The first step is a computation to find the critical Jacobian ellipsoid where this series crosses the pear-shaped series. It is found that the axes are nearly in the ratio 8 : 10 : 23, the actual values being given to five significant figures. The ellipsoidal harmonic analysis is then applied in order to find out in what way the pear-shaped figure develops from this critical case. It is shown that the principal changes take place at first in the longest axis. One end of this is lengthened and narrowed, the other shortened and blunted. In the following paper the approximation is carried to a higher order with the the result that the deviation from the ellipsoid is small everywhere except at the narrowed end and it there takes more nearly the form of a protuberance. Sir George Darwin conjectures that the further stage in the development would be the appearance of a furrow at this end, suggesting an ultimate detachment of a small portion of the mass. The general resemblance to a pear is much less striking than in M. Poincaré's conjectural figure drawn without a reduction to numbers.

The chief object of this second paper is, however, the determination of the stability of the series. Sir George Darwin comes to the conclusion that they form a stable series up to some point not determined. He is nevertheless careful to state M. Liapounoff's dissent from this conclusion in spite of the fact that a reexamination of his own work and a repetition of his calculations has not altered the coefficient by which the stability is determined.

In the last paper of volume III "On the figure and stability of a liquid satellite," the problems with which the series of memoirs originally started, as well as other kindred problems, are finally solved. The author opens with a tribute to Edouard Roche whose paper on fluid masses was published in 1847–50 at Montpellier but was neglected for many years. Roche's problem was the determination of the figure and stability of a liquid satellite revolving without relative motion in a circular orbit about a planet which is spherical and rigid.

Sir George Darwin replaces the rigid planet by a mass of fluid which can take the form appropriate for relative equilibrium, but he remarks that the analysis for both is practically the same up to the point where the stability has to be considered. He considers cases of different as well as of equal masses. In both problems the liquid assumes approximately ellipsoidal forms, so that the deformations may be conveniently determined by ellipsoidal harmonic analysis. At the limit of stability of the Roche figure, it is shown that the surfaces of the two bodies are separated by an interval which is not zero. When both the masses are liquid, the series of figures can be continued until they overlap, so that the dumbbell-shaped figure again appears as a possibility. Unfortunately, these figures before junction become very unstable and the author considers that it is inconceivable that a junction by a narrow neck of fluid should render them stable.

Thus the gap between two detached liquid masses and a single mass is not yet filled by stable figures of equilibrium. That one may not draw final conclusions concerning the genesis of a two-body system from this result is sufficiently certain. There are possibilities pointed out by the author—other figures of equilibrium, changes produced by heterogeneity of density, turbulence of motion during the fission,—not yet investigated, any one of which may furnish the desired solution of the problem.

The short paper entitled "The approximate determination of the form of Maclaurin's ellipsoid," published in the *Transactions* of the Society in 1903, is to be recommended as a beginning in a study of the methods employed to reach these conclusions. This problem, which of course admits of an exact solution, is treated by the methods employed in the more complicated cases. As the analysis is comparatively simple and is only carried to the degree of approximation necessary for clearness of exposition, it is easy to follow the several steps and to obtain a comprehensive view of the method.

One further paper on cosmogony is placed in Volume IV. Of this, Sir George Darwin says "I find it very difficult to estimate its value. If it is to be judged by the amount of comment to which a paper gives rise, it is a failure; for it has received but little notice." Perhaps an estimate fairer to its author would be that it was published (1889) somewhat ahead of the time when the need for such an investigation became pressing. Briefly, it consists of an attempt to show that under certain conditions, fully defined, the nebular hypothesis of Laplace will lead to developments in the early history of a planetary system somewhat resembling those produced by the accretion of meteoric matter. While the former of these hypotheses has received a degree of attention commensurate with the time it has been before the scientific public, the latter is comparatively young and has yet to undergo extensive criticism and development before it may take a place beside its older rival. It is not hard to believe that before long this paper may form a foundation for the comparison of the two hypotheses and that the methods used therein may serve as a basis for a reconciliation or a distinction which may lead to the general acceptance of one or the other.

When Dr. G. W. Hill published in 1877 his now famous papers on the motion of the moon, probably neither he nor his early readers imagined that they would open up a new field in the study of problems concerning the motions of particles and bodies. He showed that it was possible, under certain restrictions, to obtain a closed orbit relative to certain moving axes and he gave a series of figures of such orbits for different values of the lunar month. M. Poincaré, taking up the idea, developed it in his essay of 1891 on the problem of three bodies, under the name of the periodic solution. Sir George Darwin, however, had evidently recognized their value before this time, for he made a recommendation to the reviewer to undertake a study of Hill's papers as a beginning likely to lead to results. He himself, however, published nothing on the subject until 1897 when the long paper entitled "Periodic orbits" appeared in the *Acta Mathematica*. Dr. Hill had taken the disturbing body (the sun) of infinite mass moving uniformly round the earth at an infinite distance, and had considered the periodic motion of a satellite of the earth; he finds a portion of a single family of orbits, the moon belonging to one member of the family. Sir George Darwin took the sun to be moving uniformly in a circular orbit at a finite distance and to be of mass equal to ten times that of the earth or planet, and he studies not only the satellite orbits of the third body but also the planetary orbits which are periodic. The main difficulty arises from the adoption of so small a mass of the sun relative to the planet. The ordinary methods of development in series become inapplicable owing either to lack of or to extreme slowness of convergence. He therefore resorted to mechanical quadratures. Starting at an arbitrary place on the sun's radius vector with a given velocity, the orbit is traced piece by piece until it crosses the axis on which it started. If it proves to be reentrant, the periodic orbit is obtained; if not, the starting point is changed and a new orbit computed. After several attempts, by judging the initial point from the previous failures, an orbit is at length obtained which is exactly periodic. There is practically one unknown constant and this has to be determined in each case by the trial and error method just explained. The devices for performing this work accurately, so that errors shall not accumulate, are ingenious.

The determination of the stability is a more difficult question, since it involves the making of an harmonic analysis of the orbit, and the computation of a determinant of many rows and columns, but it is essential and is carried out in each case with complete success. Many orbits belonging to several different families are found. This is pioneer work into a region practically untouched previously and has opened the way to further investigation. In a later paper improved methods and some additional orbits are given.

The remaining papers in the two volumes cover a variety of subjects and several of them are highly suggestive. They include astronomical and cosmogonic investigations, mechanical devices, thrust of a mass of sand, the formation of ripple marks in sand, the treatment of observations, and two statistical papers (1875) on the effects of marriage between first cousins. A detailed mention of them would take us too far. There are also three addresses which one is glad to have in permanent and convenient form, as well as the English version of "The Tides" written for the Encyclopädie der mathematischen Wissenschaften.

The care which the author has bestowed on the editing of his papers has added greatly to the value of the collection. Besides the correction of small errors, calculations have occasionally been repeated and in the cases where he considers that his work has been in error, such portions are omitted and an extract or even a whole paper by another writer who may have furnished the correction is inserted. One is struck too by the generous and full acknowledgment which is given to those from whose papers, letters, or remarks the author has derived any ideas or assistance. The fourth volume completes the list of papers published up to the end of 1910: we hope that as further material accumulates it may be cast into the same form.

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MATHEMATICAL ECONOMICS.

Manuel d'Economie politique. Par VILFREDO PARETO. Paris, Giard et Brière, 1909. 695 pp.

IN the year 1906 Pareto's Manuale di Economia politica con una Introduzione alla Scienza sociale was published at Milan. Three years later a French translation by Alfred Bonnet, with revisions by the author, appeared as volume 38 in the Bibliothèque internationale d'Economie politique. This work, although written by the economist who has most insistently stood for the scientific mathematical method in economics, is not itself primarily mathematical except in spirit. The text to be sure, uses curves freely, is replete with logical keenness, and draws considerably upon mathematical language and upon mechanical, or even thermodynamical, analogy in discussing and illustrating economic equilibrium; but the strictly mathematical treatment, which might be technical to the point of causing some of the world's best literary economists insuperable difficulties, has in every instance been relegated to the long appendix of 133 pages. This arrangement has also been adopted by Irving Fisher in his recent work on The Purchasing Power of Money. So long as only a few students of political economics acquire the necessary knowledge of calculus such a segregation must remain inevitable.

In the first place, as there seems to be no very widespread notion among mathematicians, perhaps even among economists, as to what mathematical economics is and does, it may be well to define a little.

It is clear that the individual as a social unit and the state as a social aggregate require a certain modicum of mathematics, some arithmetic and algebra, to conduct their affairs. Under this head would fall the theory of interest, simple and compound, matters of discount and amortization, and, if lotteries hold a prominent place in raising moneys, as in some states, questions of probability must be added. As the state becomes more highly organized and more interested in the scientific analysis of its life, there appears an urgent necessity for various statistical information, and this can be properly obtained, reduced, correlated, and interpreted only when the guiding spirits in the work have the necessary mathematical training in the theory of statistics. (Figures may not lie, but statistics compiled unscientifically and analyzed incompetently are almost sure to be misleading,* and when this condition is unnecessarily chronic the so-called statisticians may well be called liars.) The dependence of insurance of various kinds on statistical information and the very great place which insurance occupies in the modern state, albeit often controlled by private corporations instead of by the government, makes these theories of paramount importance to our social life.

As all these subjects of study are generally treated at our universities by the departments of political economy, if indeed they are treated at all, it might be natural to classify them as mathematical economics. And one doubtless recalls that M. Cantor adopts (from Petty?) the title Politische Arithmetik for his popular little text on such matters. But the term mathematical economics as Pareto† uses it, and as it is generally used, implies something very different, something much more akin to theoretical economics, just as the term mathematical physics does not denote the use of mathematics to reduce physical observations or even to give the theory of special physical apparatus so much as it connotes general theoretical physics. It is evident to every physicist who lifts his nose from the grindstone of his lengthy experiments, who frees and rests his mind from the ofttimes exasperatingly human perversities of his instruments, that it is only with the aid of mathematics that the salient facts of physics can be adequately set forth and correctly correlated, the underlying concepts and suppositions critically analyzed and properly refined. It is evident that notwithstanding the magnificent genius of Faraday, electromagnetism needed its Maxwell and its Lorentz, that howsoever wondrously Regnault might measure, thermodynamics needed its Gibbs and its Planck.

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* Those who have followed critically the government crops reports, emanating from the Department of Agriculture, have not failed at times to complain that better and plainer results might somehow be obtained.

† We would not imply that Pareto has not been interested in statistical questions; his researches in these lines are well known.

All this is true of economics, but here it is not true that every economist sees it. So intricate are the interrelations of economic phenomena, so hopeless are any extensive experimental methods, so fragmentary are as yet the desirable statistics, and so far removed from the possibility of accurate quantitative formulation does the subject seem to be, that perhaps the majority of economists, doing their best to grope their way through a fog of details and a mist of prejudices, feel it to be impossible, or at best a useless work of supererogation, to introduce mathematical methods into economics. Now, to be sure, if economists expect of the application of the mathematical method any extensive concrete numerical results, and it is to be feared that like other non-mathematicians all too many of them think of mathematics as merely an arithmetical science, they are bound to be disappointed and to find a paucity of results in the works of the few of their colleagues who use that method. But they should rather learn, as the mathematicians among them know full well, that mathematics is much broader, that it has an abstract quantitative (or even qualitative) side, that it deals with relations as well as numbers, and that it has been defined as the science which draws necessary conclusions.*

There is no doubt that economists draw conclusions. Of these some are tenable, some are not. A few of those tenable are necessary, many are not. Indeed it is inevitable, and for a long time will remain so, that economics, especially in its applications, should be an art, and that its conclusions, far from being necessary, should largely be a matter of the opinion of the individual economist. But just because the way is dark with detail and shrouded in prejudice and impossible of complete clarification, is that any reason why any one should scorn whatever illumination mathematics can bring, whatever precision of statement it may suggest, whatever necessary conclusions it offers? If one knows precisely what his premises are and just how much he can draw from them, is he not the better off for exercising his judgment when strict deduction is no longer possible? So it seems to us, and we must admit that when we read anything that Pareto writes, whether a short article in a popular review or an essay* in the mathematical encyclopedia or a vast manual of political economy, we are agreeably impressed not alone with the cogency of the proofs, but even more with the soundness of the judgments.

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* For a discussion of what mathematics is, see M. Bôcher, "The fundamental conceptions and methods of mathematics," in this BULLETIN, vol. 11, pp. 115–135.

Two quotations may be of interest as showing how vitally economists differ as to the use of mathematics in their field.

"Mathematics has influenced the form and the terminology of the science, and has sometimes been useful in analysis; but mathematical methods of reasoning, in their application to economics, while possessing a certain fascination, are of very doubtful utility."†

"The effort of the economist is to *see*, to picture the interplay of economic elements. The more clearly cut these elements appear in his vision, the better; the more elements he can grasp and hold in his mind at once, the better. The economic world is a misty region. The first explorers used unaided vision. Mathematics is the lantern by which what before was dimly visible now looms up in firm, bold outlines. The old phantasmagoria disappear. We see better. We also see further."‡

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* We use this word advisedly; for it does not seem that Pareto's 50-page article (with apparently more to follow) in the new, French, edition of the mathematical encyclopedia is other than an essay, excellent in itself, but as an encyclopedia article on mathematical economics decidedly inferior to his earlier 25-page presentation of the same subject in the German edition. However, the whole encyclopedia, whether German or French edition, seems of late to have run riotously and fruitlessly to leaves.

† From the Encyclopedia Britannica, 11th edition, vol. 8, p. 902, in the long article on Economics which is signed W. A. S. H. As this signature is not reproduced in the key at the beginning of the volume, it is difficult to identify the author, but as this one sentence is his only reference to the vast literature of mathematical economics we are forced to believe that the editors of the new edition of the Britannica have in this instance, as in some others we have observed, let out an important subject to a man who, however eminent he may be, has not that complete command of his subject and that full sympathy with all its phases which are necessary and which alone bring competence for writing with the poise, dignity, and authority appropriate to a great encyclopedia. We should have had much more confidence in an article signed F. Y. E. On the other hand we must admit that W. A. S. H. is, in comparison with some others, extremely inoffensive and even gracious toward his mathematical confrères.

‡ From p. 119 of Irving Fisher's "Mathematical investigations in the theory of values and prices," *Transactions of the Connecticut Academy*, vol. 9 (1892), pp. 1–124. We cannot cite this work without commending it to all for the clearness and precision with which the economic problem is stated, for the keen appreciation of the limitations, as well as the advantages, of the use of mathematics in economics or other sciences, and

The foregoing paragraphs may seem rather remote from a review of the book under discussion; but they are not. For one of the things which the author makes clearest, by constant reference to it at all stages of his work, is the scientific attitude which he is taking, the attitude which he believes should be taken toward theoretical economics. He nowhere claims to do the impossible, he does not desire to overrate the usability of mathematics in applied economics, but he does insist upon an analysis, as thorough as possible, of the underlying foundations of economic theory. He is even willing to minimize the value of mathematics in the discussion of the simpler cases, where the reasoning may readily be carried out in words,* for the sake of emphasizing the fact that it is the interdependence of the totality of economic phenomena which brings in the complications necessitating the introduction of mathematics.

Let us begin now with the appendix. We consider an idealized individual, the homo œconomicus, who lives in the presence of and is possessed of certain goods and whose actions will result from his desire to obtain the greatest pleasure from the appropriation or "consumption" of the goods. If he is in possession of given amounts x, y, $\cdots$ of different goods, he will generally be equally satisfied, neither more nor less, by certain other combinations of amounts. If his possessions be represented by a point in a space of as many dimensions as the number of goods considered, the totality of the points which represent equal satisfaction will constitute a surface of one dimension less in that space. This surface is a surface of indifference, that is, of equal desire or pleasure. By considering that he could have a choice between some different combination and its totality of equivalents, another surface could be drawn. Thus the space is ruled off with a family of

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for the critical discussion of the case against mathematics as presented by its opponents. The readers of Pareto's manual and of Fisher's monograph cannot fail to remark the similarity between the scientific points of view of the two authors.

* For instance on p. 557 he says: Nous estimons que l'usage des mathématiques, pour des problèmes du genre de celui d'un individu et de deux, ou même de plusieurs biens, ne donne pas des résultats dont l'importance puisse se comparer à ceux que l'on obtient dans les cas de l'équlibre économique général. A notre avis c'est l'interdépendance des phénomènes économiques qui nous oblige à faire usage de la logique mathématique. Cette manière de voir peut être bonne ou mauvaise; mais en tout cas elle ne doit pas être confondue avec celle des économistes qui établissent des théories en négligeant précisément cette interdépendance.

surfaces of indifference

$$f(x, y, \cdots; I) = 0 \quad \text{or} \quad I = \psi(x, y, \cdots) \quad \text{or} \quad I = F[\psi(x, y, \cdots)],$$

where the parameter I may be taken as the index of pleasure or ophelimity.*

We may now pass by differentiation to the equation

$$dI = F'\psi_x dx + F'\psi_y dy + \cdots = 0,$$

which is the differential equation of the surfaces of indifference. This equation merits a comment. The negative partial derivatives such as

$$-\left(\frac{dy}{dx}\right)_{v,\ldots} = +\frac{\psi_x}{\psi_y},$$

which it defines, express the rate at which one "good" will be indifferently exchanged for another. This, however, is nothing but the price of one good relatively to another. Hence the ratios

$$\psi_x : \psi_y : \cdots = p_x : p_y : \cdots$$

are the price ratios p relative to a fixed good as base. Be it observed, however, that these are not in general the actual market prices, but the prices which the given individual would himself establish. Moreover, the actual terms like $F'\psi_x = I_x$ are the rates of increase of I on the marginal amounts dx, and so forth; that is, they are the marginal ophelimities or utilities, which are so fundamental in economics.

Now conversely, one may estimate the price ratios of indifferent exchanges and may thus set up a differential equation

$$\varphi_x dx + \varphi_y dy + \cdots = 0$$

of which the integral would be I or a function of I. This, however, implies that such an integral exists. The discussion of the existence of this integral was suggested as a necessity by Volterra in commenting on the Italian edition of the manual.†

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* The functional symbol F may be taken more or less arbitrarily according to the scale on which I is estimated; for although the value of I may be distinguished from a greater or a smaller value, the estimation of I is hardly quantitative.

† The same suggestion was made already in 1892 by Fisher, loc. cit., p. 88, on mathematical grounds; but I believe that considerations indicated by Gibbs have since led him to abandon the suggestion.

The author now gives a careful treatment of the question, which is necessarily somewhat complicated by the every-day fact that the consumption of certain amounts, dx, dy, $\cdots$ of the various goods may yield very different increments of pleasure according to the order in which they are consumed—not many would care to reverse the order of their dinner menu. (Pareto, page 251).

Now this reference to a menu is exceedingly liable to cause confusion; for the order of consumption which is in question as regards the integrability of the above differential is merely a conceptual order or curve in the space of the goods x, y, $\cdots$, whereas the order of consumption which is involved in the menu is a time order which either has nothing to do with the other order or has some very fundamental and subtle and neglected relation to the whole question of utility. Suppose that we could integrate the equation and thus obtain an index of pleasure; this index ought, it would seem, to apply either at a given instant (and to contain the time as a parameter, owing to the fact that at different times the price ratios and integrating factor are different), or it ought to apply to an interval of time such as a day or a year during which conditions were supposed to remain the same (and the time would then not enter at all). Fisher in his memoir distinctly takes the latter point of view, and indicates its advantages in smoothing out capricious elements, whims instead of well balanced desires. This is the time-average way of obtaining definite results. But the homo œconomicus may well be considered an idealized averaged individual free from caprice. This is the ensemble-average way of getting definiteness. Pareto seems to us to take this point of view. (Students of statistical mechanics are acquainted with the two types of average and with their essential identity for many problems.) But in neither case does the time enter into the determination of the equilibrium or of the conceptual path followed to attain the point of maximum pleasure—unless, indeed, the whole analogy with mechanics and thermodynamics is to break down in a vital point. We must remember that in statics the time element is ignored and that in thermodynamics changes are supposed to take place infinitely slowly. The difference in the increase of pleasure which a person would experience in eating his dinner in inverse as opposed to direct order cannot, we believe, be logically taken into account, and we are inclined to think that the whole discussion of the order of integration, a point that naturally occurs to the mathematician, can be thrown out of court by the economist. For on the ensemble-average basis with an instantaneous comparison of the desirability of increments of different goods, there is no period of time during which they can be consumed in temporal order, the "consumption" is mere conceptual appropriation; whereas on the time-average basis the consumer is free to consume in the temporal order he chooses. The author in places comes pretty near saying this himself; he might have decided to say it if he had observed that to eat one's dinner in the proper order with too little leisure or with too lengthy waits between the courses would be quite as bad as to have to eat it in the wrong order, for we suppose he would hardly desire to include the intervals of time themselves as consumed goods so as still to have different orders.

We have spoken of the exchange ratios or price ratios which the given individual would regard as fitting his desires. These are not in general the market prices, or proportional to them. Hence the homo œconomicus will avail himself of this divergence to make exchanges which will increase his value of I, and he will pursue these exchanges until his private exchange ratios become identical with those of the market. He will thus come into equilibrium with the market with those particular amounts $x, y, \cdots$ of goods which will maximize I subject to the imposed constraints. For example, if he starts with the amounts $x_0, y_0, \cdots$ and if the market prices are constant,* his exchanges must satisfy the equation

$$(x - x_0)p_x + (y - y_0)p_y + \cdots = 0,$$

and his price ratios must satisfy the $n - 1$ equations

$$\varphi_x : \varphi_y : \cdots = p_x : p_y : \cdots.$$

Thus there are just n equations wherefrom to determine the final or equilibrium values of the n quantities $x, y, \cdots$. The problem is determinate.

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* Although in the last analysis it is the desires of all which regulate the prices in the market, it is permissible as a first approximation to many phenomena to regard these prices as constant for any one individual; his purchases and sales are insufficient materially to change the market quotations.

This simple problem in the determination of a point of equilibrium is typical of the various more complicated problems of the same sort. The analysis really amounts to formulating the equations, counting the number of unknowns, and checking it with the number of equations. To many modern mathematicians the fear that the equations might be either redundant or incompatible would probably be so strong as to deter them from seeing much of value in the analysis. But it should be remembered that not so very long ago the method of counting constants was widely used in pure mathematics even though the science was then much more highly developed toward arithmetic equations than is now the case with economics. Moreover, in a physical science the question of rigor is very different from that in mathematics; to be ultra-rigorous mathematically may be to be infra-rigorous physically. To throw out Gibbs's phase rule because its proof, being essentially a count of constants, is no proof at all would be equally good mathematics and equally bad physics. On the other hand, setting up exact mathematical relations, compatible and uniquely soluble, over the whole range of variation of the variables might be a wonderful mathematical tour de force while being viciously misleading physics in the neighborhood of certain critical points where a slight change of the variables introduces such wide variations in the functions as to make the problem just as indeterminate physically as it has become determinate mathematically.

It would not do, however, to give the impression that the use of analysis in economics is restricted to counting constants. It is necessary to study the general properties of the surfaces of indifference, and this leads to the quadratic differential form for the second differential d^2I. The law of demand and supply, in the simple case of a single individual in relation to the whole market, requires partial derivatives and transformations among them which are well comparable to those required in thermodynamic theory. If space permitted, we should be glad to stop at this point and sketch the proof of the law that the demand decreases as the price rises. It would perhaps be still more indicative of the usefulness of mathematics if we could run through the author's analysis of Marshall's hypothesis that for small variations of price and of quantities the marginal utility of money could be treated as constant; for Pareto shows by an extended series of reductiones ad absurda, which apparently would be difficult to establish without some use of mathematics, that the hypothesis is not tenable. Further on, in the discussion of production, which is probably the most complicated in the whole appendix, it becomes necessary to maximize integrals, and the author is therefore led to introduce into his analysis the ordinary algorisms of the calculus of variations.

It is thus seen that it is not merely the mathematics of a first course in calculus which may be used to advantage in economic theory. Interesting as it would be to pursue these matters in detail, we feel that we must come back to the main text and offer some short account of it. In the first chapter on general principles the author first analyzes the different aims which may lie behind the study of political economy and carefully delimits himself to the single one of developing theoretical economics. He then turns his attention to discussing the relation of a theory to the actual phenomena of which it is the theory. Although he is writing with his own subject in view, this introductory essay is conceived and carried out in such a broad spirit that it may serve almost equally well as a sort of prolegomenon to the study of any scientific theory, and as such it can be recommended for general reading.

Chapter II, an introduction to social science, is delightful reading and forms a sort of amplification of the preceding chapter upon the social and moral side. Honest and manly in its views, pithy and pungent in its epigram, it calls to mind the direct diction and unrelenting exposition of W. G. Sumner's Folkways. Pareto, however, did not make use of, and perhaps did not have at hand, the mass of detailed references to folkways which Sumner adduced to drive in and rivet in his opinions. For this reason those who are especially interested in this chapter should supplement their study of it by reading the Folkways. Indeed if economics is a misty region under a pall of prejudices, where fierce war is often waged around the banners of certain words—trusts, protection, supply and demand, single tax—of the meaning of which the combatants have not always a very clear perception, even more is this true of social science, and one who would really attain a free and independent judgment in this field has to exercise the greatest care. One of the most interesting observations of the author is (page 122) that there is an antagonism between the conditions of action and of knowledge, that faith alone drives men to action, and that it is not for the good of society that very many persons should be occupied with the scientific study of social questions.*

It is the next four chapters, Notion générale de l'équilibre économique, Les goûts, Les obstacles, L'équilibre économique, which contain for the most part that portion of the text which is supplemented by the mathematical appendix, and which is most scientific in its development. It is this part which seems so different from the older treatises on economics, not only on account of the generous use of curves as in a number of modern texts, but particularly because of the great care with which the problems arising are divided into separate sub-problems and the exhaustive manner in which the various sub-cases are treated. We have tried to give an idea of economic equilibrium in a simple case; for any idea of the multifarious distinctions which arise in the presence of various degrees of free competition and monopoly we must refer to the text. But to show how conscientious the author is in exhibiting real as well as theoretical phenomena, we will cite an instance or two. It is a statement often found in texts on economics that the more we have of a thing, the less valuable to us are equicrescent increments of it; that is, the marginal utility of anything decreases with an increasing amount of it. And many an author states this as a general law. Pareto, however, makes clear the fact that there are exceptions. It does no harm to amplify the matter even further, to point out that in this life our desires consist of a lot of little ones and a few big ones, that to satisfy the latter large increments of certain goods (say, of money) are necessary, and that therefore the curve of marginal utility of money consists in reality of long slow descents (in the intervals where the little desires are better and better satisfied, but the satisfaction of one of the large ones seems hopelessly remote) interrupted by sharp ascents (in the intervals where the fulfilment of one of the greater desires seems imminent). In another place the author, when discussing the proposition that the returns on capital differently invested ought in a state of equilibrium to be the same, cites some very interesting statistics concerning Belgian corporations. It appears that from 1873 to 1887 there were founded 1,088 companies with a paid-in capital of 1,493 millions of francs. Up to 1901 the sum of 515 millions had been paid back under liquidation, and the remaining companies were earning 56 millions a year. This is only 5.7 per cent. on the 978 millions remaining in the original investment. Although the figures are only rough, it is certain that the return on the 1,088 enterprises could not have much exceeded the current rate of interest. In this connection let us remark that some student statistician looking for a thesis might consider the problem of calculating as accurately as possible the rate of return on capital variously invested in this country.

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* A footnote appended to these remarks may be of interest. Par example le livre de M. Roosevelt, American Ideals, pourra peut-être servir à pousser à l'action les citoyens des Etats-Unis, mais assurément il n'ajoute rien à nos connaissances, et sa valeur scientifique est très voisine de zéro. Pareto never minces matters in the expression of his opinion, and to a careless or prejudiced reader might therefore appear unjust or bitter when he meant to be only frank.

Chapter VII is on various topics connected with the population—the distribution of wealth, relations between economic conditions and the population, malthusianism, stability and selection, and so on. In the next chapter the author takes up capital, and treats savings, interest, exchanges, international commerce, bimetalism, and the like. His definition of capital appears to us considerably less definite than that of Irving Fisher in his Capital and Income and Rate of Interest.* Indeed it may be safe to hazard the guess that Pareto, who seems to esteem Fisher's work very highly, might have made considerable use of it if only it had appeared early enough to have been easily available for the Manuel. It goes without saying that when treating such subjects as population and capital, the author is merging into the realms of applied economics where the argument has to run somewhat differently from its previous course.

In the final chapter, Le phénomène économique concret, we have, among many other things, a delightful essay on present conditions and tendencies in our economic or social life. To any one who is perplexed at our rapid social evolution, who is confused in his ideas of trusts or of labor unions, who is bewildered at the way in which the rising costs of government are bleeding the bourgeoisie for the support of the lower classes, to any one who wonders where we are and whither we are going, no better reading can be suggested than this chapter of Pareto's; it is short enough to keep the attention, virile enough to claim the admiration, of the most nonchalant reader. Much as the author's scientific economics appeals to us, we must say that his sturdy vision into society and social evolution are even more to our liking; and there can be little doubt that for many of us, mathematicians or otherwise, the most inspired portions of Pareto's Manuel will be the sociology of Chapters II and IX.

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* Reviewed in this BULLETIN, vol. 15, p. 169.

In closing this review it may be only proper to apologize a bit for its tardiness. At the same time we may perhaps point out that it is now while we are in the midst of modifying constitutions, whether actually or under various disguises such as preferential primaries or judicial rule of reason, while we are attacking or about to attack the formerly sacred institution of private property, under many a pretext other than the real one that might makes right, now while we in America are perhaps more than ever conscious of our present or impending social evolution is the time when we may derive the most benefit from having our attention called to Pareto's observations on the ills that beset us. As for the mathematical economics, that is still good, and still will be good at any time when we may settle down to study it.

EDWIN BIDWELL WILSON.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY.

NOTES.

THE programme of the Fifth international congress of mathematicians at Cambridge, England, begins on Wednesday evening, August 21, with a reception of the members in the Hall of St. John's College. The opening meeting will be held on Thursday morning and the afternoon will be devoted to organisation, election of officers, and two of the special lectures. On the following days sectional meetings will be held in the morning and lectures will be delivered in the afternoon. The congress will divide into four sections: I. Arithmetic, algebra, analysis; II. Geometry; III. Mechanics, physical mathematics, applications; IV. Philosophical, historical, and didactic questions. In connection with Section IV three discussions inaugurated by the International commission on the teaching of mathematics will be held. The sections will be further subdivided as the number of papers may require.

The following special lectures are announced: M. Bôcher: "Boundary problems in one dimension." E. Borel: "Définition et domaine d'existence des fonctions monogènes uniformes." E. W. Brown: "Periodicity in the solar system." F. Enriques: "I problemi relativi ai principii della geometria." B. Galitzin: "The principles of instrumental seismology." E. Landau: "Gelöste und ungelöste Probleme aus der Theorie der Primzahlverteilung und der Riemannschen Zeta-funktion." J. Larmor: "The dynamics of radiation." W. H. White: "The place of mathematics in engineering practice."

Members of the congress unaccompanied by ladies can be accommodated in the colleges at a fixed payment of 6*s*. per day for rooms, attendance, breakfast, and dinner. Luncheon will be provided at 1*s*. 6*d*. Ladies will be received at Newnham College at 6*s*. per day for rooms, attendance, and meals. Members desiring accommodation in the Colleges are requested to notify the general secretary, Professor E. W. Hobson, Christ's College, before August 1. The fee for membership in the congress is £1. The family of a member are entitled to the privileges of membership, except the receipt of a copy of the Proceedings, on payment of 12*s*. each.

At the meeting of the London mathematical society held on April 17 the following papers were read: By H. S. Carslaw "An application of the theory of integral equations to the equation $\nabla^2 u + k^2 u = 0$"; by A. Cunningham, "On Mersenne's numbers."

The following university courses in mathematics are announced for the year 1912–1913:

University of Chicago. All courses are four hours a week.—By Professor E. H. Moore: Integral equations in general analysis; General seminar on mechanical quadrature, continued fractions, and boundary problems (throughout the year).—By Professor G. W. Myers: History of mathematics (winter).—By Professor L. E. Dickson: Theory of invariants (autumn and winter); Theory of numbers (winter); Theory of equations and linear algebras (spring).—By Professor J. W. A. Young: Limits and series (winter).—By Professor H. E. Slaught: Differential equations (autumn).—By Professor G. A. Bliss: Theory of functions (autumn); Definite integrals and abelian integrals (winter); Hyperelliptic functions (spring).—By Professor E. J. WILCZYNSKI: Selected topics in geometry (autumn and winter); Projective differential geometry (spring).—By Professor A. C. LUNN: Graphical analysis and theory of attraction and potential (autumn); Fourier series and Bessel's functions and vector analysis (spring).

CLARK UNIVERSITY.—By Professor W. E. STORY: Analytic geometry of higher plane curves, higher surfaces, and twisted curves, three hours; Calculus of operations and finite differences, three hours; Theory of errors, three hours first half-year; Infinitesimal geometry, three hours second half-year; Seminar.—By Professor H. TABER: Theory of functions, five hours; Integral equations, two hours first half-year; Hypercomplex number systems, two hours second half-year; Seminar.—By M. DE PEROTT: Theory of numbers, two hours first half-year; Abelian integrals, two hours second half-year.

HARVARD UNIVERSITY.—By Professor W. E. BYERLY: Advanced calculus, three hours; Dynamics of a rigid body, three hours; Trigonometric series, introduction to spherical harmonics and the potential function, three hours, with Professor B. O. PEIRCE.—By Professor W. F. OSGOOD: Advanced algebra, three hours (second half year); Theory of functions, second course, three hours.—By Professor M. BÔCHER: Ordinary linear differential equations, three hours.—By Professor C. L. BOUTON: Elementary theory of differential equations, three hours (first half year); Geometric transformations, three hours.—By Professor E. V. HUNTINGTON: Fundamental concepts of mathematics, three hours (second half year).—By Professor J. L. COOLIDGE: Introduction to modern geometry and modern algebra, three hours; Geometry of the circle, three hours.—By Professor G. D. BIRKHOFF: Theory of functions, first course, three hours; Calculus of variations, three hours (first half year).—By Dr. D. JACKSON: Infinite series and products, three hours (first half year); Definite integrals, three hours (second half year).

Various courses in reading and research are also offered on special topics, and Professors OSGOOD and BIRKHOFF will conduct a fortnightly seminar in the theory of functions.

UNIVERSITY OF ILLINOIS.—By Professor E. J. TOWNSEND: Complex variables, three hours.—By Professor G. A. MILLER:

Elementary group theory, three hours.—By Professor H. L. Rietz: Actuarial theory, three hours (first term).—By Professor C. H. Sisam: Differential geometry, three hours.—By Professor J. B. Shaw: Fourier series, three hours.—By Professor A. Emch: Elliptic functions, three hours.—By Dr. A. R. Crathorne: Linear differential equations, three hours.—By Dr. R. L. Börger: Modern algebra, three hours.—By Dr. E. B. Lytle: History of mathematics, three hours.

Indiana University.—By Professor S. C. Davisson: Ordinary differential equations (a, w), three hours; Fourier's series (s), three hours; Theory of functions (a, w, s), two hours.—By Professor D. A. Rothrock: Advanced calculus (a, w, s), three hours; Higher geometry (a, w), three hours.—By Professor U. S. Hanna: Theory of errors (a), three hours; Substitution groups and Galois theory (w, s), three hours.—By Professor R. D. Carmichael: Functions of an infinite number of variables (a, w), three hours; Partial differential equations (a, w), three hours; Theory of numbers (s), five hours; Seminar in difference equations (a, w, s), two hours. (a, w, s=autumn, winter, spring quarters.)

Johns Hopkins University.—By Professor F. Morley: Higher geometry, two hours; Dynamics, two hours (second term); Seminar, two hours.—By Professor A. B. Coble: Theory of correspondences, two hours; Theory of probabilities, two hours (second term).—By Professor A. Cohen: Theory of functions, two hours; Differential equations, two hours (first term); Theory of numbers, two hours (second term).—By Mr. H. Bateman: Integral equations, two hours.

Massachusetts Institute of Technology.—By Professor H. W. Tyler: Elements of the theory of functions, two hours.—By Professor F. S. Woods: Partial differential equations, two hours.—By Professor F. H. Bailey: Fourier series, two hours.—By Professor E. B. Wilson: Advanced calculus and differential equations, three hours; Hydrodynamics, one hour; Theoretical mechanics, two hours; Mathematics of wireless telegraphy, two hours.—By Dr. H. B. Phillips: Thermodynamics, two hours.—By Dr. F. L. Hitchcock: Curve tracing, two hours; Elementary theory of the potential function, two hours.

Princeton University.—By Professor H. D. Thompson: Analytic geometry, three hours; Infinitesimal geometry, three hours.—By Professor L. P. EISENHART: Differential geometry, three hours; Mechanics, three hours.—By Professor O. VEBLEN: Algebra, three hours; Seminar, three hours.—By Professor J. G. HUN: Analytic projective geometry (second term), three hours.—By Professor E. SWIFT: Calculus of variations (second term), three hours.—By Professor J. H. McL. WEDDERBURN: Theory of functions of a complex variable, I (first term), three hours.

THE following courses are announced for the summer session of 1912:

UNIVERSITY OF CHICAGO.—All courses are four hours a week.—By Professor E. H. MOORE: Integral equations, first term; General analysis, first term.—By Professor E. J. WILCZYNSKI: Synoptic course in advanced mathematics; Higher geometry.—By Professor A. C. LUNN: Graphical analysis; Units and dimensions; Vector analysis.—By Professor E. R. HEDRICK: Definite integrals; Theory of functions of a complex variable.—By Professor G. A. MILLER: Finite groups.

COLUMBIA UNIVERSITY.—By Professor JAMES MACLAY: Theory of functions of a complex variable.—By Professor M. W. HASKELL: Modern analytic geometry.—By Professor W. B. FITE: Theory of groups of finite order.

CORNELL UNIVERSITY.—All courses are five times a week.—By Professor V. SNYDER: Projective geometry.—By Professor W. B. CARVER: Teachers' course in geometry.—By Dr. C. F. CRAIG: Differential equations.—By Dr. F. W. OWENS: Teachers' course in algebra.

INDIANA UNIVERSITY.—By Professor S. C. DAVISSON: Ordinary differential equations, five hours, first half-quarter.—By Professor D. A. ROTHROCK: Advanced calculus, five hours; Higher plane curves, three hours.—By Professor R. D. CARMICHAEL: Linear differential equations, five hours; Infinitesimal analysis, three hours; Difference equations, two hours.

PROFESSOR G. KOWALEWSKI, of the technical school at Prague, has accepted a professorship of mathematics at the German University of Prague.

DR. R. COURANT has been appointed docent in mathematics at the University of Göttingen.

DR. HOSTINSKY has been appointed docent in mathematics at the Bohemian University of Prague.

DR. F. PFEIFFER has been appointed docent in mathematics at the technical school at Danzig.

PROFESSOR P. APPELL, of the University of Paris, has been elected a corresponding member of the academy of sciences of St. Petersburg.

PROFESSOR E. CZUBER, of the University of Vienna, has been elected to membership in the academy of sciences of Halle.

PROFESSOR A. DEMOULIN, of the University of Ghent, has been elected a corresponding member of the royal society of Liège.

PROFESSOR G. A. MILLER, of the University of Illinois, has recently been elected corresponding member of the Sociedad matemática Española of Madrid.

PROFESSOR P. P. BOYD, of Hanover College, Indiana, has accepted a professorship of mathematics at the Kentucky State University.

PROFESSOR H. B. FINE, of Princeton University, has been granted leave of absence during the academic year 1912–1913, to study and travel in Europe.

MR. W. W. DENTON and DR. M. W. CHITTENDEN have been appointed instructors in mathematics at the University of Illinois.

MR. C. A. BARNHART, of the University of Illinois, has been appointed professor of mathematics at Carthage College, Carthage, Illinois.

DR. R. K. MORLEY, of the University of Illinois, has been appointed assistant professor of mathematics at the Worcester Polytechnic Institute.

DR. G. F. MCEWEN, of the University of Illinois, has accepted a position as hydrographer for the biological station of the University of California.

MR. A. S. GALAJIKIAN, of Cornell University, has been appointed assistant professor of mathematics at the University of the Philippines.

MR. L. C. COX, of State College, Pennsylvania, has been appointed instructor in mathematics at Purdue University.

DR. J. R. CONNER, of the Johns Hopkins University, has been appointed associate in mathematics at Bryn Mawr College.

DR. J. I. TRACEY has been appointed instructor in mathematics in the academic department of Yale University.

DR. R. M. WINGER has been appointed instructor in mathematics at the University of Illinois.

MR. H. BATEMAN, of Bryn Mawr College, has accepted a teaching scholarship at the Johns Hopkins University.

MR. W. V. LOVITT, of the University of Washington, has been appointed instructor in mathematics at Harvard University; Mr. A. L. MILLER has been advanced to an instructorship in mathematics.

PROFESSOR A. E. HAYNES, of the University of Minnesota, has retired after thirty-eight years continuous service as a teacher.

PROFESSOR S. W. SHATTUCK, of the University of Illinois, will retire at the end of the present academic year. He has been connected with the department of mathematics since the founding of the University in 1868.

PROFESSOR M. C. ARZELÀ, of the University of Bologna, died at St. Stephen di Magra, March 16, 1912, at the age of 65 years. He was a member of the Academy dei Lincei and of the national Italian society of sciences.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ADHÉMAR (R. d'). Leçons sur les principes de l'analyse. Tome 1: Séries. Déterminants. Intégrales. Potentiels. Equations intégrales. Equations différentielles et fonctionelles. Paris, Gauthier-Villars, 1912. 8vo. 6+324 pp. Fr. 10.00

BACHELIER (L.). Calculs des probabilités. Tome 1. Paris, Gauthier-Villars, 1912. 4to. 7+518 pp. Fr. 25.00

CALVET-AZAL. Essai sur la notion de quantité imaginaire. Paris, Gauthier-Villars, 1912. 8vo. 48 pp.

CERVALLO (E.). Le calcul des probabilités et ses applications. Paris, Gauthier-Villars, 1912. 8vo. 9+171 pp. Fr. 6.50

LIEBMANN (H.). See MARKOFF (A. A.).

MANGOLDT (H. v.). Einführung in die höhere Mathematik. 2ter Band: Differentialrechnung. Leipzig, Hirzel, 1912. 8vo. M. 15.40

MANNING (H. P.). Irrational numbers and their representation by sequences and series. New York, Wiley. 12mo. 6+123 pp. Cloth. $1.25

MARKOFF (A. A.). Wahrscheinlichkeitsrechnung. Nach der 2ten Auflage des russischen Werkes ubersetzt von H. Liebmann. Leipzig, Teubner, 1912. 8vo. 7+318 pp. Cloth. M. 13.00

MEISSNER (O.) Wahrscheinlichkeitsrechnung nebst Anwendungen. (Mathematische Bibliothek, herausgegeben von W. Lietzmann und A. Witting.) Leipzig, Teubner, 1912, 8vo. 4+64 pp. M. 0.80

MÜHE (A.). Beweis des Fermatschen Prinzips. 2te, erganzte Auflage. Berlin, Reimer, 1912. 8vo. 4 pp. M. 0.75

II. ELEMENTARY MATHEMATICS.

BARLOW. Table of squares, cubes, square roots. New York, Spon, 1911. 12mo. 200 pp. $1.50

BERTRAND (G.). Trattato di algebra elementare. Prima traduzione italiana, con note ed aggiunte di E. Betti. Nuova edizione, per cura di A. Socci. 15a impressione. Firenze, Le Monnier, 1912. 16mo. 7+531 pp. L. 3.00

BETTI (E.). See BERTRAND (G.).

BORCHARDT (W. G.). Junior algebra examples. London, Rivingtons, 1912. 8vo. 198 pp. 2s.

HENSING (K.) Geometrie. 3tes Heft. Leipzig, Hoffmann, 1911. 8vo. pp. 115–254. M. 1.60

HUNTINGTON (E. V.). Four place table of logarithms and trigonometric functions. Unabridged edition. London, Spon, 1912. 3s.

LESSER (O.). See SCHWAB (K.).

LINNICH (M.). See SCHWAB (K.).

MILNE (W. J.). Key to first year algebra. New York, American Book Co., 1912. 12mo. 288 pp. $0.80

ROPP (C.). Calculator and short-cut arithmetic. 10th edition, revised and enlarged. Chicago, Ropp, 1912. 8vo. 192 pp. $1.25

SALOMON (A.). Leçons d'algèbre. 7e édition. Paris, Vuibert, 1912. 16mo. 267 pp. Fr. 2.00

SCHMEHL (C.). Lehrbuch der Stereometrie für höhere Lehranstalten. Giessen, Roth, 1912. 8vo. 6+160 pp. M. 2.50

SCHWAB (K.) und LESSER (O.). Mathematisches Unterrichtswerk. Fur hohere Mädchenschulen bearbeitet von M. Linnich. 1ter Teil. Wien, Tempsky, 1911. 8vo. 143 pp. Cloth. M. 2 00

——. Geometrie. Bearbeitet von M. Linnich. 2ter Teil. Leipzig, Freytag, 1912. 8vo. 184 pp. Cloth. M. 2.50

SOCCI (A.). See BERTRAND (G.).

TABLES de logarithmes à cinq décimales et pour les fonctions trigonométriques. Par F. I. C. Edition double: Division sexagésimale et division centésimale. Paris, Gigord, 1912. 16mo. 8+255 pp.

TESTI (G. M.). Elementi di matematica. Fascicolo II. 5a edizione corretta. Livorno, Giusti, 1912. 16mo. 7+78 pp. L. 0.80

WALLENTIN (F.). Maturitätsfragen aus der Mathematik. Auflösungen. 6te Auflage. Wien, Gerold, 1912. 8vo. 6+236 pp. Cloth. M. 4.50

III. APPLIED MATHEMATICS.

ABRAHAM (M.). Theorie der Elektrizität. 1ter Band: Einfuhrung in die Maxwellsche Theorie der Elektrizität. Mit einem einleitenden Abschnitte von A. Föppl. 4te, umgearbeitete Auflage, herausgegeben von M. Abraham. Leipzig, Teubner, 1912. 8vo. 18+410 pp. Cloth. M. 11.00

ANNUAIRE pour l'an 1912, publié par le Bureau des Longitudes. Avec des notices scientifiques. Paris, Gauthier-Villars. Fr. 1.50

BAKKER (G.). Théorie de la couche capillaire des corps purs. Tome II: La couche capillaire des corps purs et ses phases homogènes et adjacentes. (Collection Scientia.) Paris, Gauthier-Villars, 1912. 8vo. 80 pp. Boards. Fr. 2.00

BIGOURDAIN (G.). See DELAMBRE (J. B. J.).

BUCHHOLZ (H.). See KLINKERFUES (W.).

CHAPMAN (S.). The kinetic theory of a gas constituted of spherically symmetrical molecules. London, Dulau, 1912. 4to. Sewed. 2s. 6d.

CHINI (M.). Corso speciale di matematiche, ad uso dei chimici e dei naturalisti. 2a edizione, con aggiunte e modificazioni. Livorno, Giusti, 1912. 8vo. 12+297 pp. L. 4.00

CRANDALL (C. L.). Textbook on geodesy and least squares. New York, Wiley. 8vo. 10 + 329 pp. Cloth. $3.00

DALLWITZ (W. v.). Wärmelehre in Theorie und Anwendung. 1ter Band: Wärmetheorie und ihre Beziehungen zur Technik und Physik. Berlin, Volckmann, 1912. 8vo. 18+331 pp. Cloth. M. 11.25

DELAMBRE (J. B. J.). Grandeur et figure de la terre. Ouvrage augmenté de notes, de cartes, et publié par les soins de G. Bigourdain. Paris, Gauthier-Villars, 1912. 8vo. 8+402 pp. Fr. 15.00

DELSOL (M. E.). Note sur le vol des oiseaux. Paris, Gauthier-Villars, 1911. 8vo. 22 pp. Fr. 1.00

DITTMANN (E.). Berechnung elektrischer Leitungsnetze. Strelitz, Hittenkofer, 1911. M. 5.00

ESPITALLIER et CHASSERIAUD (R.). Cours d'aviation. Livre I: Appareils d'aviation et propulseurs. Paris, Gauthier-Villars, 1911. 4to. 295 pp. Fr. 13.00

FÖPPL (A.). See ABRAHAM (M.).

FRISCHAUF (J.). See SOLDNER (J.).

GRAMONT (A. de). Essais d'aérodynamique. 2e série. Paris, Hachette, 1912. 4to. 115 pp. Fr. 3.50

GROSSMANN (H.). Agenda 1912 pour l'horlogerie, la bijouterie et la petite mécanique. 9e année. Paris, Gauthier-Villars, 1911. 32mo. 372 pp. Fr. 2.00

GUILHAUMON (J. B.). Astronomie et navigation suivies de la compensation des compas. 5e édition. Paris, Berger-Levrault, 1912. 8vo 615 pp. Fr. 10 00

HAUCK (G.). Vorlesungen uber darstellende Geometrie. Herausgegeben von A. Hauck. 1ter Band. Leipzig, Teubner, 1912. 8vo 12+339 pp. Cloth. M. 12.00

HOSMER (G. L.). Azimuth. New York, Wiley, 1909 16mo 5 + 73 pp. Morocco. $1.00

INGALLS (J. M.). Interior ballistics. 3d edition. New York, Wiley, 1912 8vo. 221 pp Cloth. $2.50

JEANS (J. H.). The mathematical theory of electricity and magnetism. 2d edition London, Cambridge University Press, 1911. 15s

KING (W. I.). The elements of statistical method. New York, Macmillan, 1912. 12mo. 245 pp. Cloth. $1.50

KLINKERFUES (W.). Theoretische Astronomie. Neubearbeitung von H. Buchholz. 3te, verbesserte und vermehrte Auflage. Braunschweig, Vieweg, 1912. 8vo. 38+1070+11 pp. Cloth. M. 50 00

KOENEN (M.). Grundzuge für die statische Berechnung der Beton- und Eisenbetonbauten. 4te, neubearbeitete Auflage. Berlin, Ernst, 1912. M. 2.00

KRIEMLER (C.). Einfuhrung in die energetische Baustatik. Berlin, Springer, 1911. M. 2.40

KÜSTER (F. W.). Logarithmische Rechentafeln fur Chemiker, Pharmazeuten, Mediziner und Physiker. 12te, neuberechnete Auflage. Leipzig, Veit, 1912. 8vo. 107 pp. Cloth. M. 2.40

LONEY (S. L.). An elementary treatise on statics. London, Cambridge University Press, 1912 8vo. 402 pp. 12s

LUCAS (A.). See MASSENET (G.).

MASSENET (G.) et LUCAS (A.) Eléments de la théorie du navire. Paris, Challamel, 1912. 8vo. 6+120 pp.

MAYER (R.). Die Mechanik der Warme. 2te Abhandlung Herausgegeben von A. von Oettingen. (Ostwalds Klassiker Nr. 180) Leipzig, Engelmann, 1911. M 1.60

MEIER (K.). Mechanics of heating and ventilation, with charts for calculation and examples. New York, McGraw-Hill, 1912 8vo. 9+161 pp $5.00

MERBACH (P. A.). Studien uber die Mechanik der magnetischen Erscheinungen. Hamburg, Hephaestos-Verlag, 1912. 8vo. 18 pp. M 0 60

MERRIMAN (M.). Elements of mechanics. Forty lessons for beginners in mechanics. New York, Wiley. 12mo 172 pp. Cloth. $1 00

——. Treatise on hydraulics. 9th edition, revised and reset. New York, Wiley, 1912. 8vo. 10 + 565 pp. Cloth. $4.00

—— and JACOBY (H. S.) A textbook on roofs and bridges Part III: Bridge design. 5th edition, partly rewritten New York, Wiley, 1912 8vo. 8 + 414 pp Cloth $2.50

MILLER (E. F.) and others. Problems in thermodynamics and heat engineering. New York, Wiley, 1911. 8vo. 3+67 pp. $0.75

MULLER (E.). Technische Uebungsaufgaben fur darstellende Geometrie. 4tes Heft. Wien, Deuticke, 1911. M. 1.25

NERNST (W.). See POLLITZER (F.).

OCAGNE (M. d'). See SECO DE LA GARZA (R.).

OETTINGEN (A. von). See MAYER (R.).

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THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and fifty-eighth regular meeting of the Society was held in New York City on Saturday, April 27, 1912. The attendance at the two sessions included the following fifty-two members:

Mr. J. W. Alexander, Professor R. C. Archibald, Mr. H. Bateman, Mr. A. A. Bennett, Professor W. J. Berry, Professor G. D. Birkhoff, Professor E. W. Brown, Professor B. H. Camp, Dr. Emily Coddington, Professor F. N. Cole, Dr. G. M. Conwell, Professor J. L. Coolidge, Professor E. S. Crawley, Dr. H. B. Curtis, Dr. L. S. Dederick, Dr. L. L. Dines, Mr. E. P. R. Duval, Professor H. B. Fine, Professor T. S. Fiske, Professor W. B. Fite, Professor O. E. Glenn, Mr. G. H. Graves, Professor C. C. Grove, Professor L. A. Howland, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Dr. J. K. Lamond, Dr. N. J. Lennes, Professor W. R. Longley, Dr. H. F. MacNeish, Dr. Emilie N. Martin, Dr. E. J. Miles, Dr. H. H. Mitchell, Professor C. L. E. Moore, Dr. R. L. Moore, Professor W. F. Osgood, Mr. E. S. Palmer, Dr. Anna J. Pell, Professor A. D. Pitcher, Dr. H. W. Reddick, Professor R. G. D. Richardson, Dr. J. E. Rowe, Professor L. P. Siceloff, Mr. C. G. Simpson, Professor D. E. Smith, Professor Sarah E. Smith, Dr. W. M. Smith, Professor Henry Taber, Professor J. H. Tanner, Professor Oswald Veblen, Miss M. E. Wells.

The President of the Society, Professor H. B. Fine, occupied the chair. The Council announced the election of the following persons to membership in the Society: Miss S. R. Benedict, Smith College; Mr. C. E. Fisher, Rhode Island Normal School; Dr. T. H. Gronwall, Chicago, Ill.; Mr. L. A. Hopkins, University of Michigan; Dr. A. J. Kempner, University of Illinois; Mr. V. C. Poor, University of Michigan; Mr. R. B. Stone, Harvard University; Mr. K. P. Williams, Princeton University. Seven applications for membership in the Society were received.

It was decided to hold the annual meeting of the Society this year at Cleveland, Ohio, in affiliation with the American Association for the Advancement of Science. The winter meeting of the Chicago Section will be merged in the annual meeting. Owing to President Fine's absence abroad, the delivery of his presidential address will be postponed to the annual meeting of 1913.

The following papers were read at the April meeting:

(1) Dr. R. L. Moore: "Concerning Jordan curves in nonmetrical analysis situs."

(2) Dr. J. K. Lamond: "Improper multiple integrals over iterable fields."

(3) Professor L. A. Howland: "Binary conditions for singular points on a cubic."

(4) Professor B. H. Camp: "Certain integrals containing parameters."

(5) Dr. S. Lefschetz: "On the V_3^3 with five nodes of the second species in S_4.

(6) Dr. E. R. Marshall: "A labor-saving device in computation."

(7) Professor G. D. Birkhoff: "The reducibility of maps."

(8) Professor G. D. Birkhoff: "A determinant formula for the number of ways of coloring any map."

(9) Professor Oswald Veblen: "An application of modular equations in analysis situs."

(10) Dr. H. B. Phillips and Professor C. L. E. Moore: "A geometric use of matrices."

(11) Dr. H. B. Phillips and Professor C. L. E. Moore: "A theory of linear distance and angle."

(12) Professor L. P. Siceloff: "Sylow subgroups in groups whose orders are of certain special forms."

(13) Professor A. D. Pitcher: "Concerning the continuity and convergence of functions of a general variable."

(14) Professor W. R. Longley: "Proof of a theorem due to Picard."

(15) Mr. A. R. Schweitzer: "Remark on a functional equation."

(16) Mr. A. R. Schweitzer: "Theorems on functional equations."

(17) Dr. Dunham Jackson: "On approximation by trigonometric sums and polynomials (second paper)."

(18) Dr. N. J. Lennes: "Concerning Van Vleck's nonmeasurable set."

(19) Dr. N. J. Lennes: "Concerning infinite polygons and polyhedrons."

The papers of Dr. Lefschetz, Professor Pitcher, Mr. Schweitzer, Dr. Jackson, and Dr. Lennes were read by title. Dr. Lefschetz's paper appeared in the May BULLETIN. That of Professor Longley is published in the present number of the BULLETIN. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In the *Transactions* of this Society, volume 6, pages 83–98, Professor Veblen has given a definition of a simple plane curve lying in a plane which satisfies Axioms I–VIII, XI of his system of axioms for geometry. Let V denote such a plane. Dr. Moore wishes to show that:

I. There is a one to one correspondence between the points of V and a part H of an ordinary number plane, this correspondence being (in an easily defined sense) continuous. Thus each point of V is represented by a pair of coordinates (x, y).

II. A simple curve as defined by Veblen may be represented by a pair of equations $x = f_1(t)$, $y = f_2(t)$, where $f_1(t)$ and $f_2(t)$ are continuous functions of the numerical parameter t.

III. The plane V is an Ebene as defined by Hilbert in his article "Ueber die Grundlagen der Geometrie" (*Mathematische Annalen*, volume 56, page 383).

2. Dr. Lytle has shown that the fundamental relation

$$\underline{\int}_{\mathfrak{A}} f \leqq \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f \leqq \overline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} f \leqq \overline{\int}_{\mathfrak{A}} f,$$

where the integrals are proper integrals, holds for a class of fields which he calls iterable. Using a new definition of an improper multiple integral, which is due to Professor Pierpont, and considering unlimited functions defined, in general, over iterable fields, Dr. Lamond gives conditions under which the above fundamental inequalities hold. In some cases the function f may have points of infinite discontinuity at an everywhere dense set.

3. The cubic plane curve has been exhaustively studied by use of the theory of ternary forms. Conditions necessary and sufficient for the existence of one or more double points, of a triple point, etc., have been derived in terms of ternary in-

variants, chiefly by Aronhold, Gordan, Clebsch, and Gundelfinger.

The theory of binary forms, however, may also be applied to the study of the cubic. Compare, for example, the treatment of the inflexions of a general cubic by Clebsch in his Theorie der binären Formen. In Professor Howland's paper conditions for the existence of a double point (node or cusp) and that of a triple point are derived, expressed in terms of the simultaneous invariants of two binary forms.

4. In a recent paper (*Annales de la Faculté de Toulouse*, series 3, volume 1 (1909), pages 25–128) Lebesgue has studied the integrals

$$\int_a^b f(t)\phi(t, n)dt, \qquad \int_a^b f(t)\phi(t - x, n)dt.$$

His fundamental theorems have been generalized by Professor Camp so as to apply to functions of m variables and k parameters, defined over limited fields of very general natures. Moreover, corresponding theorems may be obtained for certain classes of functions not considered by Lebesgue. Applications of the results are found in the theory of the development of an arbitrary function in series of normal functions.

6. In insurance computations multiplications or divisions of a series of numbers by a constant are of frequent occurrence, in which the results are taken to the nearest unit, or 1st, 2d, or 3d, etc., decimal. A new method for obtaining the results of such serial multiplication or division with a minimum of labor is set forth in Dr. Marshall's paper. The method is based upon the following consideration:

Let $x_{0,1}, x_{0,2}, \cdots, x_{0,k}, \cdots; x_{1,1}, x_{1,2} \cdots, x_{1,k}, \cdots; \cdots, x_{i,1}, x_{i,2}, \cdots, x_{i,k}, \cdots; \cdots$ represent the required quotients, where the first subscripts indicate their nearest integers, and let $a_{0,1}, a_{0,2}, \cdots$ represent the corresponding variable dividend, while m is the constant divisor, so that $x_{i,k} = a_{i,k} \div m$. Then we have $x_{0,k} < .5 \leqq x_{1,k} < 1.5 \leqq x_{2,k} < 2.5 \cdots$.

Hence, in general, $a_{i,k} < (i + .5)m \leqq a_{i+1,k} < (i + 1.5)m$ (where the $=$ of the double symbol may hold only for $k = 1$). Hence all the values of $a_{i+1,k}$ giving the integer $i + 1$ lie between $(i + .5)m$ and $(i + 1.5)m$. For multiplication the reciprocal of m, obtained to a certain degree of approximation, is used instead of the constant multiple m itself.

The author illustrates the process by a concrete example from insurance practice, and proves that on the arithmometer, which is an especially adapted calculating machine for this kind of computation, the saving of labor resulting from the use of this method is between 75 and 90 per cent. In the second part of his paper the author shows how to extend his method to the finding of quotients and products in which an approximation of a number of decimals is required and proves by actual illustration that his method can be advantageously used in a large field of actuarial practice, comprising divisions with moderately large differences of results, not exceeding two or three figures. It should be stated that the method is of most advantage only in cases where the differences of the results are smaller than those of the given series of multipliers or dividends. The illustrations of the second part of the paper are taken from Mr. George King's conversion and valuation tables appended to his paper, "On policies with deferred participation in profits, etc.," read before the Faculty of Actuaries in Scotland and printed in the *Transactions* of the Faculty, volume 5, part 9, number 83 (1911).

In conclusion the author shows how his method could also be used with considerable advantage in the computation of deferred and temporary annuities.

7. It is well-known that the problem of coloring a given map in four colors is reducible to that of coloring other maps of fewer regions in case the given map has (1) any multiply connected regions; (2) more than three regions meeting at any point; (3) any region bounded by less than five others; these facts comprise nearly all that has been done toward the reduction of maps.

By defining reducibility in a technical sense in which a principle due to Kempe is involved, Professor Birkhoff shows that numerous other configurations occur only in reducible maps; thus any ring of less than five regions and a ring of five regions not bounding a single region are proved to occur only in reducible maps.

8. This note by Professor Birkhoff on a formula for the

number of ways of coloring a map in any number of colors will appear in the *Annals of Mathematics*.

9. In the note by Professor Veblen it is shown that the circuits and systems of circuits in a complex of lines (in the sense of analysis situs) may be represented by the solutions of a system of linear equations, modulo two. The operations of linearly combining these equations or their solutions are interpreted geometrically. A similar theory holds for complexes of n dimensions. The equations may be applied in various problems of analysis situs, including the four color problem and the proof of Euler's theorem.

10. A point A in n dimensions is determined by $n + 1$ homogeneous coordinates and may be represented by the matrix $\|a_1 \cdots a_{n+1}\|$. A linear function $\lambda A + \mu B$ of two points is represented by the one-rowed matrix whose elements are $\lambda a_i + \mu b_i$. A line AB is represented by a two-rowed matrix $[AB]$ formed by writing the coordinates of A above the coordinates of B. The elements of this matrix are the two-rowed determinants obtained from it and not the individual coordinates. The sum of two such matrices is formed by adding the corresponding elements. This new matrix is not, in general, expressible as a two-rowed matrix of $n + 1$ columns. Similarly, planes are represented by three-rowed matrices $[ABC]$, three-spaces by four-rowed, etc.

These quantities $[AB]$, $[ABC]$, etc., may be considered as products obeying the laws of Grassmann's outer product, and are called progressive. The operation consists of writing the coordinates of the first factor in the first row, the coordinates of the second factor in the second row, etc., to form a new matrix.

Hyperplanes are used as coordinates in the same way and the products thus obtained are called regressive. If the number of points representing the factors of a product is greater than $n + 1$, the factors are expressed in plane coordinates and multiplied regressively. One of the chief aims of Dr. Phillips and Professor Moore is to express this regressive product in terms of points, thus obtaining Grassmann's reduction formula.

11. In this paper Dr. Phillips and Professor Moore define

the distance between two points, in three dimensions, by the following properties: (1) If one point is fixed and the distance held constant, the other describes a plane. (2) Distances between points on a line have the additive property. It is found that the distance is expressible in terms of a fundamental system consisting of a plane and a linear complex. Distance will then be invariant under a six-parameter group of collineations leaving the fundamental system fixed. The angle between two planes is defined as the dual of the distance between two points. In this case the fundamental system consists of a point and linear complex. In order that distances and angles be left invariant by the same six-parameter group of collineations, the complexes are taken as the same and the point as the pole of the plane with respect to the complex. In terms of this fundamental system angle between lines, distance from point to line, etc., are defined. Formulas for the area of a triangle and volume of a tetrahedron are also obtained. These results are generalized for n dimensions.

12. In this paper Professor Siceloff studies restrictions on the structure of a group corresponding to certain restrictions on the form of its order. He proves in the first theorem that a group of order $(kp + 1)p$ cannot have the full number, $kp + 1$, of subgroups of order p if p is greater than some prime-power factor of $kp + 1$. The second theorem states that in a group whose order has only one factor (other than unity) $\equiv 1 (\text{mod } p)$, the elements common to two Sylow subgroups of order p^a form an invariant subgroup. The third theorem deals with an important special case under the hypothesis of theorem 2.

13. In his Introduction to a Form of General Analysis and in his lectures at the University of Chicago Professor E. H. Moore has given a theory of developments of a general range and has indicated the beginning of a general theory of continuity and convergence of functions relative to a development of the range of the functions. Professor Pitcher secures the ordinary properties of continuous and convergent functions by means of simple postulates on the development of the range of the functions.

15. The purpose of Mr. Schweitzer's first paper is to explain

a reference to *Crelle's Journal* given on page 300 of his paper, "On a functional equation," BULLETIN, volume 18, pages 299–302. The relation $\phi(x,y) = \Omega\{\psi(x) - \psi(y)\}$ can be written $\Omega^{-1}\{\phi(x,y)\} = \psi(x) - \psi(y)$, and the preceding reference suggests $\Omega^{-1}\{x\phi(y) - y\phi(x)\} = \psi(x) - \psi(y)$ as a correlative of the equation of Abel, *Crelle*, volume 2, page 389. Corresponding to Abel, l. c., page 386, one gets a particular case of the relation $\chi\{\phi(x,y)\} = \chi(x) - \chi(y)$ by assuming that $\Omega^{-1}(x) = \psi(x)$ in the preceding correlative. It is of interest to note here that C. Lottner, *Crelle*, volume 46, pages 367, 368, etc., has discussed a special case of $\Omega^{-1}\{\phi(x,y)\} = \psi(x) + \psi(y)$ in relation to the preceding paper of Abel.

16. In connection with his recent article in the BULLETIN, March, 1912, page 300, Mr. Schweitzer proves the following theorems:

I. If $\phi[x, \phi(y, z)] = \phi[y, \phi(x, z)]$ and $\phi[\phi(y, z), z] = y$ and $\phi[\phi(y, z), y] = z$, then there exists a function χ_ϕ such that $\phi(x,y) = \chi_\phi^{-1}\{\chi_\phi(x) + \chi_\phi(y)\}$ and $\phi(x,y) = \chi_\phi^{-1}\{\chi_\phi(x) - \chi_\phi(y)\}$.

II. If $\phi[y, \phi(x, z)] = \phi[z, \phi(x, y)]$ and $\phi[\phi(y, z), z] = y$ and $\phi[\phi(y, z), y] = z$, then there exists a function χ_ϕ such that $\phi(x, y) = \chi_\phi^{-1}\{\chi_\phi(x) - \chi_\phi(y)\}$ and $\phi(x,y) = \chi_\phi^{-1}\{\chi_\phi(x) + \chi_\phi(y)\}$.

17. Following out further the line of investigation on which he reported at the February meeting, Dr. Jackson is led to the following conclusions: If $f(x)$ is a function of period 2π which satisfies a Lipschitz condition with coefficient λ, that is, if it is always true that

$$|f(x_2) - f(x_1)| \leqq \lambda |x_2 - x_1|,$$

then $f(x)$ can be approximately represented for all values of x by a trigonometric sum of order n or lower with a maximum error not exceeding $3\lambda/n$, for all positive integral values of n. If $f'(x)$ satisfies a Lipschitz condition with coefficient λ, the maximum error of such an approximation for $f(x)$ can be made not to exceed $20\lambda/n^2$. If $f(x)$ satisfies the same conditions, except as to periodicity, in a closed interval of length 1, it may be represented in this interval by a polynomial of the nth or lower degree with an error not greater than $\frac{3}{2}\lambda/n$ or $10\lambda/n^2$ respectively. If $f(x)$ satisfies the first of the conditions above, it differs from the partial sum of its Fourier series, to terms of the nth order, by not more than $6\lambda \log n/n$, provided $n \geqq 5$.

18. In this paper Dr. Lennes describes in detail the construction of a set which is non-measurable in the sense of Lebesgue. The general method is the same as that used by Van Vleck (*Transactions*, volume 9, page 237), but each step is made to depend upon explicitly formulated postulates.

19. In two papers published in the *American Journal of Mathematics*, volume 33, Dr. Lennes formulated a certain body of theorems on polygons and polyhedrons. With one exception these were confined to figures having a finite number of sides or faces. In the present paper many of the results obtained in these papers are extended to polygons and polyhedrons having an infinite number of sides or faces.

F. N. COLE,
Secretary.

PROOF OF A THEOREM DUE TO PICARD.

BY PROFESSOR W. R. LONGLEY.

(Read before the American Mathematical Society, April 27, 1912.)

CONSIDER the ordinary differential equation of the first order and second degree

$$(1)\qquad Ap^2 + 2Bp + C = 0 \qquad (p = dy/dx),$$

in which the coefficients are power series in x and y vanishing when $x = y = 0$

$$(2)\qquad A = ax + a_1y + \cdots,\quad B = bx + b_1y + \cdots,\quad C = cx + c_1y + \cdots.$$

Picard has proved that in the general case* every integral curve (real curve in the cartesian plane) of equation (1) which comes infinitely near the origin, actually reaches the origin with a determinate tangent. The proof† is based upon the existence of an analytic integral curve passing through the origin, a condition which is satisfied in the general case. But such a curve does not always exist, and the following proof, which makes no use of its existence, gives more information concerning the form of the integral curves near the origin.

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* By the general case it is meant that there exists no particular relation of equality among the coefficients of (2).

† Picard, *Comptes Rendus*, vol. 120 (1895) p. 524; *Math. Annalen*, vol. 46 (1895), p. 521; Traité d'Analyse, vol. 3, pp. 217–225.

To examine the behavior of integral curves in the neighborhood of the origin we may limit the coefficients in equation (1) to terms of lowest degree and we suppose that A, B, and C contain only first degree terms in x and y. Solving equation (1) for p gives

$$(3) \qquad p = \frac{-B + \sqrt{\Delta}}{A},$$

where $\Delta = B^2 - AC$. Through every point at which Δ does not vanish there are two integral curves with distinct tangents depending on the choice of the sign of the radical. Suppose this choice has been made. Then the form of equation (3) shows that the slope depends only on the ratio y/x and hence the slope of an integral curve crossing the line $y = mx$ is the same at every point of the line. From this it follows that it is possible to trace a continuous integral curve by making the radius vector from the origin turn always in the same direction, provided the curve does not reach a point at which $\Delta = 0$ where the sign of the radical could be changed.

The form of the integral curves depends on the nature of the discriminant locus, the equation of which is $\Delta = 0$, and there are two cases to be considered: (1) when the discriminant locus has a conjugate point at the origin, and (2) when the discriminant locus consists of two distinct branches through the origin.

In the first case Δ vanishes for $x = y = 0$ and is positive for all other values of x and y. Hence if an integral curve does not pass through the origin it may be followed by making the radius vector rotate always in the same sense about the origin. It is conceivable that an integral curve might have the form of a spiral approaching the origin asymptotically. To do this the slope must change sign four times during one revolution of the radius vector; twice by passing through zero (at the highest and lowest points) and twice by passing through infinity (at the extreme right and left points). To see that this is impossible we consider equation (3). The lines $A(ax + a_1y = 0)$, $B(bx + b_1y = 0)$, and $C(cx + c_1y = 0)$ pass through the origin and by a rotation of the axes the line B may be taken for the X-axis. The line C is the locus of points where the slope of integral curves is zero. Starting at any point near the origin on the positive X-axis with a definite sign for the radical we may follow an integral curve by making the radius vector rotate always in a counterclockwise direction. If the slope changes sign by passing through zero when the curve crosses the line C above the X-axis, then p will not change sign when the same line is crossed below the X-axis, because the sign of the radical remains the same while the sign of the quantity B is reversed. Hence the curve can not meet the positive X-axis a second time and the spiral is impossible.

In the second case the discriminant locus has two distinct branches passing through the origin. In one of the angles between the branches Δ is positive and there exist real integral curves, while in the other angle Δ is negative and there are no real integral curves. By a transformation of coordinates we may take the two branches of the discriminant locus for the coordinate axes so that Δ is positive in the first and third quadrants and negative in the second and fourth quadrants. Since $\Delta = B^2 - AC$ is positive in the first quadrant and negative in the second it is evident that the lines A and C must pass through the origin with positive slope, that is, they must lie in the first quadrant. In this case an integral curve can not have the form of a spiral, but it is conceivable that it might oscillate between the coordinate axes an infinite number of times approaching the origin. To see that this is impossible suppose that we choose a definite sign for the radical in equation (3) and begin to follow an integral curve from a point on the positive X-axis. The radius vector must rotate in a counterclockwise direction until the Y-axis is reached. To follow the curve further the sign of the radical must be changed at this point and the radius vector turned clockwise towards the X-axis. Each time the curve traverses the first quadrant it must cross the line A once and the line C once. In crossing the line C with one sign for the radical the slope of the curve will change sign by passing through zero, while with the opposite sign for the radical the slope of the curve will not change sign in crossing C. Similarly in crossing the line A with one sign for the radical the slope will change sign by passing through infinity, while with the opposite sign for the radical the slope will not change sign. Hence in starting from the X-axis, crossing to the Y-axis, and returning to the X-axis the slope of an integral curve must change sign twice and only twice; once by passing through zero (at a maximum or minimum point) and once by passing through infinity (at an extreme right or extreme left point).

Suppose now an integral curve Γ leaves the X-axis with positive slope. In order to reach the Y-axis the slope must become negative when Γ crosses A. Then in crossing C (either the first or second time) the slope will change to positive by passing through zero, and with a positive slope Γ can not return to the X-axis.

Suppose Γ leaves the X-axis with negative slope and meets A first. In order to reach the Y-axis it must cross A the first time without changing to positive slope. After having crossed C twice the curve will return towards A with positive slope and can not again meet A because at the second point of intersection the slope of Γ would have to be infinite. Hence Γ can not return to the X-axis.

Suppose Γ leaves the X-axis with negative slope and meets C first. (*a*) If the slope does not change sign then Γ, after crossing A twice, will return towards C with positive slope and can not meet C again because at a second point of intersection the slope would have to be zero. (*b*) If the slope changes to positive at the first intersection with C then the curve could not cross A twice since the slope would have to be infinite at one of the points of intersection. Hence Γ can not return to the X-axis.

The preceding argument establishes the theorem as given by Picard. No mention has been made of the case in which the discriminant locus has a cusp at the origin. If the coefficients in equation (1) are limited to terms of the first degree, a cusp at the origin means that Δ is a perfect square and the equation can be broken up into linear factors. In this case the spiral is a possible form for integral curves, but this conclusion does not follow when the higher degree terms are present unless the cusp degenerates into two coincident branches.

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A TREATISE ON PHYSICS.

Traité de Physique. Par O. D. CHWOLSON. *Ouvrage traduit sur les éditions russe et allemande* par E. DAVAUX. *Edition revue et considérablement augmentée* par l'Auteur, *suivie de Notes sur la Physique théorique* par E. et F. COSSERAT. Tome 1: *Mécanique. Etats de la matière. Acoustique.* vii + 1092 pp. 1908. Tome 2: *L'Energie rayonnante.* 1188 pp. 1909. Paris, A. Hermann.

DESPITE the fact, and perhaps for the very reason that teachers of the first general collegiate course in physics, which usually falls in the sophomore year, are widely and chronically of the opinion that there exists no text suited to their needs, there are upon the market a great variety of elementary general treatises on physics, varying from a moderate volume of three or four hundred pages to lengthy works of three or four times the size, like that old standby Ganot. When we come to more advanced compendia, suitable for the maturer student of physics, we have not many to choose from, and none, so far as we recall, in the English language. Indeed it might well be questioned whether, with the present extended and rapidly growing domain of physics, it would be possible for a single author properly to cover the whole ground, whether it might not be necessary to fall back upon collaborations as in Winkelmann's Handbuch der Physik. We may remember that over ten years ago A. Gray printed a first volume of a Treatise on Physics,* which in its masterly manner gave promise of being just what was needed in this line; but apparently the work has been abandoned, much to our disappointment. However, Chwolson has succeeded in carrying a similar task, monumental as it is, to an eminently satisfactory conclusion.

So long as Chwolson's work remained in the original Russian it could be of little use outside of Russia. But as surely as a first class work appears the Germans, often in the guise of the house of B. G. Teubner, are apt to find it out and to publish a translation; this time, however, it was the firm of Vieweg und Sohn who rendered the German and international public the service of printing Chwolson's treatise in German. That

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* Reviewed in this BULLETIN, volume 8, p. 403.

Hermann should continue the international conquest of the work is testimony not only to its high value, but to his own courage; for the French seem none too prone to take kindly to foreign texts, and the undertaking of a four volume treatise on physics, to run into thousands of pages and to sell at about 150 francs, is no light tax on one's faith and one's resources. It is with great pleasure that we observe that the demand for the work has been so great as already to necessitate a second edition, at least of the earlier fascicules.* As the French edition is the latest, it is also the best; for the author has been able to revise and elaborate. The translator has added some notes, and E. and F. Cosserat have furnished additional contributions.

The work commences with a general introduction which shows that Chwolson is somewhat of a philosopher. It is probably true that everybody who has large views on a wide range of phenomena must have some sort of philosophy which is interesting to him to expound and interesting to others to read, but which in the last analysis is rather a personal matter with which almost everybody else may differ. For instance the author states that the more limited the number of hypotheses, the more advanced the state of science. This is a beautiful ideal that appears to have little realisation in practice. From the opposite point of view one might just as well hold that the more advanced the state of science, the greater the diversity of hypotheses, and the better known the limitations or field of application of each. The hypothesis of imponderable caloric has never disappeared, and probably never will, from the theory of the flow of heat. To be compelled to treat heat as molecular motion in this case would be entirely too inconvenient. Some physicists now desire to abolish the hypothesis of an ether. They cannot do it; the best they can do is to formulate a new hypothesis to organize phenomena which are not satisfactorily organized under the ether-hypothesis. But as our philosophy cannot be as valuable to our readers as the author's should be to his, we should be lacking in humor if we persisted.

An elaborate discussion of physical quantities or magnitudes of different sorts and of units (not the theory of dimensions) is a valuable contribution to the introduction. This is a subject which gives many students considerable trouble, but should not worry readers of this work. The notion of density, for example, is used in very different ways in physics, and needs the careful treatment the author gives. We must observe, however, that for all his care he defines ordinary density as it is ordinarily defined, namely, as a limit, whereas in reality on the molecular hypothesis no such limit can exist. This is but another instance of the diversity of hypothesis; no one disputes the molecular view of matter, yet few take the precaution to observe that the contradictory hypothesis of a continuum is constantly being used. We have no objection to this plurality of hypotheses, but we would not slur it over.

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* What was tome 1, fasc. 1, of 407 pages in 1906 has now become in 1911 tome 1, vol. 1, considerably larger. (I am reviewing from the earlier edition, which has only recently come to me for that purpose. E. B. W.)

The exposition of the relative fields of experimental, theoretical, and mathematical physics is lucid. It is doubtful, however, whether the mathematical method does not encroach more upon and is not more necessary to theoretical physics than seems to be implied on page 11; and indeed the work as a whole seems to give more space to mathematical physics* than one might imagine from the introductory statements. It should be pointed out that mathematics is used in physics not merely to deduce consequences from well established facts, but to formulate new hypotheses and enquire after their consequences. In this latter sense much mathematical physics is as truly experimental as is considerable laboratory work.

The introduction closes with some mathematical work on differentiation, curvature, and vector addition. This is necessarily very meager and will be sufficient only for the more elementary parts of the text, some portions of which require very considerable mathematical training and ability. As references to original sources are very numerous throughout the remainder of the work, and form a very valuable part of it, the author inserts here a list of the most important periodicals cited, with their appropriate abbreviations.

Some elementary mechanics comes next—a little kinematics, rectilinear and curvilinear, the analysis of inertia, mass, and force according to Newton's laws, momentum, impulse, resultants, couples, center of inertia, and moment of inertia. The principle of the conservation of mass is not mentioned without adding a list of references to various recent experiments attempting to detect changes (not, of course, the very recent experiments in connection with moving charges and the principle of relativity). The author states that there is a very general notion that Newton's second law refers to rate of change of momentum, and that he cannot share that opinion, but regards the law as dealing with acceleration. This same point of view is taken by Jouguet in his searching historical study of mechanics.* The English attitude, however, has seemed to incline more to the rate of change of momentum; and it may be interesting to note that it was by going back to the definition of force as rate of change of momentum that G. N. Lewis was able in studying relativity to assign a definite mass to a particle, instead of the varying masses from transverse to longitudinal that previous writers had used.

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* Due mainly to additions by translator and editors, and unfortunately of the French non-physical type.

The author refers for the determination of centers of inertia to earlier and more elementary studies of the reader; in finding moments of inertia he uses triple integration without apology, except to say that readers unacquainted with the method may take the results for granted. When he goes on to treat work and energy, he uses sums, limits of sums, and simple integrals, even line integrals, without hesitation. We cannot help wondering how necessary it is to develop the small amount of differential calculus above mentioned if one is to go right on with integral signs. It seems as though it might have been better either to give a considerable account of mathematical analysis or to have taken the whole of elementary calculus for granted. In studying energy, the various forms (not merely the kinetic and potential energies of mechanics) are taken up and the three principles, namely, that the energy is a determinate function of the state, that energy is conserved, and that it may possess a direction of transformation, are elaborated.

The treatment of harmonic motion is detailed, including the composition of vibrations, damped oscillations, propagation of a disturbance by longitudinal or transverse waves, wave surface and ray, Huyghens's principle, reflexion, refraction, diffraction, and Doppler's principle. As the material covers over 60 pages, it is readily seen to be sufficiently emphasized. The author next proceeds to study universal gravitation and the newtonian potential with the minimum amount of mathematics, parabolic and pendulum motion in the field of gravity, and the theory of dimensions as applied to mechanics. There seems to be no need of any other comments than these: The treatment of the ordinary mechanics of particles and rigid bodies seems very meager and the use of mathematics extremely limited for a work of this size, which itself subsequently gives considerable development to the mechanics of continuous media. In this particular, the work appears unbalanced and inferior to Gray's Treatise on physics. We should recommend the addition of some theoretical mechanics with Gray as a model, although it would not be necessary to carry the matter so far as he does. The bibliography on mechanics is extensive and well chosen; but in regard to this list, and most others in the course of the work, we have been wholly unable to see why the author or editor did not make the arrangement according to date or alphabetical according to author or title instead of tossing the references together apparently pell-mell or following the order in which they were cited in the text.

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* E. Jouguet, Lectures de Mécanique, reviewed in this BULLETIN, vol. 18, p. 32, without mentioning this particular point.

We have said that as far as mechanics is concerned Chwolson's work is unbalanced; but we will hasten to add that this may not be due so much to his own doings as to his polite complacency in accepting the additions of others. For right here (pages 236–273) we come upon a long note by the Cosserats upon the dynamics of a point and of a rigid body, which has absolutely no place in this work, and should have had much less space even if the author had developed his mechanics to include Lagrange's equations and Hamilton's principle. We postpone further comment.

The following 125 pages are given to instruments and methods of measurement. To one not at all an experimental physicist this makes clear, pleasant, and instructive reading. There is good general advice as to methods of measurement, the elements of least squares, formulas of approximation, numerous diagrams of instruments designed to make mechanical (as distinguished from thermal or electric) measures, and ample explanation. Even the experiments to determine the mean density of the earth are discussed. (Why Burgess is not mentioned in this connection we do not understand.) There is a very appropriate note on the theory of integrating machines by the translator.

The second fascicule of Volume I deals with the gaseous state. If we are not mistaken, the author's choice of hydrogen (H = 1) as standard instead of oxygen (O = 16) is now somewhat old-fashioned. The experiments necessary for testing the equation $pv = RT$, and the results obtained, are naturally the first. In future editions of the book the work of P. W. Bridgman, who has succeeded in running pressures from 3,000 up to 12,000 atmospheres, will have to be cited with Natterer's, Cailletet's, and Amagat's on page 429. Clapeyron's formula and van der Waals's equation, with some of the modifications that have been suggested, are merely touched at this point. There follows a chapter on barometers, airpumps, and similar apparatus.

After some mention of the conduct of gases in the presence of other gases or liquids or solids (laws of Dalton, Henry, etc.), Chwolson takes up the kinetic theory and, while not going into very complicated mathematical analysis, is quite able to show the relations of the theory to the previously developed laws, and to discuss the mean velocity, mean free path, viscosity (Maxwell's law), and van der Waals's equation. Sutherland's formula for the viscosity is given without going into his theoretical deduction. Indeed it is characteristic of the author's good sense and balance to run through the development of such an important theory as the kinetic, and then add without any theoretical derivation some formula which has been found to fit the experimental data better than the one derived from that theory. The interesting chapter on the motion of gases (efflux, diffusion, resistance to projectiles, dissociation, etc.) is rendered somewhat unintelligible in our copy by the fact that pages 537–552 are omitted and 521–536 repeated through an error in binding, but enough can be seen to show that the interesting work of Hugoniot and recent formulas by Boussinesq are not overlooked.

A fascicule of over 300 pages is given to the properties of the liquid and solid state. For the most part the presentation is descriptive rather than mathematical or theoretical. Surface tension and capillarity receive considerable attention. The theory of electrolytic dissociation is mentioned, but the facts about solutions are here sketched without much attention to that theory, which is reserved for a later volume. In regard to osmotic pressure the author likewise refrains from adopting definitely any hypothetical explanation, contenting himself with emphasizing the difficulty of finding a satisfactory explanation by calling attention to several of those which have been suggested. Friction and viscosity are briefly discussed.

The treatment of the motion of liquids is partly descriptive, as might be found in a work on hydraulics, and partly mathematical, due perhaps to various additions by the translator. This makes the chapter read somewhat unevenly. Indeed it seems almost useless to include even the small amount of the theory of vortices here given unless the reader has had more thorough training in the composition of angular velocities than was suggested in the early work on mechanics. Moreover it is rather unfortunate, from the molecular point of view, to use such an expression as "the two molecules are always upon a vortex line" (page 711). It may be wondered whether this is Chwolson's own language.

The chapter on the colloidal state has evidently been written or at least edited by V. Henri, which is sufficient recommendation for it.

In regard to the treatment of the solid state there seems to be no need of comment. The same unevenness between the extremes of mere description on the one hand and considerable difficult mathematics of elasticity on the other is noticeable; indeed there are parts of the latter which would still seem of rather doubtful utility in this work even if the more extended treatment which we should recommend for mechanics were given.

In a great treatise on physics, like this, mechanics, hydromechanics, and elasticity should be introduced just so far as they develop and illustrate fundamental physical principles either in themselves or in their applications. The ideas of momentum, moment of momentum, and energy cannot be too highly emphasized, especially in view of Maxwell's theory of electromagnetics and modern theories on radiant energy. We should introduce the gyroscope (not here mentioned, so far as we observe*) because there is a lot of physics in it that can hardly be so well found elsewhere. We should discuss Lagrange's coordinates and Hamilton's principle because without these it is difficult to get the idea of a general dynamical system and of the most important single principle in mathematical (may we say theoretical?) physics. We should study hydrodynamics and elasticity for the physical principles involved, for the conceptions connected with dynamics in a medium, to emphasize the analogies and points of difference between liquid and solid media, the ether, and discrete systems.

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* Except in connection with the gyrostatic theory of light!

We should bring to light and use general principles of reciprocity, and the like. But we should be rather wary of introducing mathematical developments which are devoid of general physical significance even though they lead to formulas of engineering or laboratory practice—unless we are frankly intent on including a good account of mathematical (as distinguished from theoretical) physics. It would not do to say that in the present edition of Chwolson's Physics there is too much mathematical work; but it is true that the physical side of a question seems almost immediately lost forever as soon as the analysis begins. This is a difficulty that is almost always in evidence in a French, as contrasted with an English, text; and if we rightly detect the uniformly physical spirit in which Chwolson himself writes, we cannot help feeling that the unevenness and mathematicity are not his.

These remarks apply also to a number of sections in the final fascicule (Acoustics) of the volume. What propriety is there in the sectional heading "Le problème général de l'acoustique. Conditions aux limites" in a work whose author in his introduction states that it is theoretical, not mathematical, physics that is important? It is hard to believe that boundary value problems really interest him. Still, although we do not feel that the translator and editors have made the best selection of additional mathematical material or have incorporated it in the smoothest fashion, we do not wish to give the impression that there may not be many readers of the work who will find these additions helpful and some who may find them the most helpful part of the treatise. There will be all sorts of readers with every diversity of requirement; and the editors probably had some particular class of readers in mind. Moreover, we have a feeling, and perhaps the translator had it, that purely mathematical physics is so widespread that some concession to it should be made. The average density of mathematics in the treatment of sound appears greater than in previous fascicules.

Although the usual topics are taken up, the author has space for a number of things not ordinarily mentioned, the propagation of finite displacements, explosive waves, where the velocities exceed the canonical. On the other hand, although diffraction of sound and the question of hearing are both treated, we do not find any mention of the way in which we perceive the direction from which the sound comes. It is interesting to note that in the section on longitudinal vibrations of strings the references are to the recent (1906) work of H. N. Davis. It is sometimes surprising how thoroughly the author has covered the literature up to the date of printing; it is by no means infrequent to see references dated within two years of the issue of the fascicule in which they are quoted. Apparently, however, the ten year old work of E. W. Scripture,* which would interest some students of sound, is not cited.

Volume I ends without an index; so does the next volume. If the complete work carries no index, or even if it carries only a poor one, a large amount of the usefulness of the treatise will be lost.

The second volume has the title Radiant Energy—a brave title. And the first fascicule displays a subheading Emission and absorption of radiant energy—a stalwart start. The complete adoption of the Maxwell theory, the firm belief in the reality of the ether and in its importance in the explanation of nature, the relentless grounding of the student in the use of a medium are characteristics of Chwolson's work which are more in evidence than in most other books and which rank his as a distinctly modern treatise.

And yet we of 1912 must admit that we cannot use the titles Radiant energy and Emission and absorption of radiant energy with the whole hearted confidence that we could wish, or that we could fairly assume five, or at any rate ten, years ago. We are in the unfortunate transitional stage of knowing too much and yet too little about radiant energy, its absorption, and its emission. We know too much to make the facts fit the Maxwell theory as we today understand it; we know too little or have too little genius as yet to found a new theory. There is a vast array of evidence that radiant energy appears usually, if not always, in discrete particles; there is a tendency to abandon the ether.

Chwolson should not be blamed for not taking toward the ether that attitude which has not yet ripened thoroughly and which was scarcely budding when the translation of his book began; rather should he be blamed if he had adopted it in place of the straight Maxwell theory, which even yet is none too clearly appreciated by the students to whom the author is talking. We cannot, however, withhold an adverse comment from the attempt to determine the density of the ether as about 10^{-17}; there are arguments (in England) just as valid for a density greater than the reciprocal of this; the whole subject is too far from settled to be given an apparently final conclusion in a text like this.

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* Yale Psychological Studies (1899), and his elaborate volume in the Yale Bicentennial Series (1901).

Much as we might desire to carry our review of this second volume along with the detail granted to the first, a detail which the work fully merits, we must refrain on account of the lengths to which it would lead. Moreover, the author has not been so often nor so vitally interrupted in his exposition by mathematical additions of the editors, and the text therefore appears so much more natural and homogeneous that comment is not so necessary. Finally, this volume on radiant energy and optics cannot readily and safely be judged apart from the subsequent presentation of electromagnetic theory in the fourth volume which we have not yet seen in completed form. We shall therefore run on rather rapidly, stopping merely to point out the general course of the development.

The start is strictly physical, with its description of the ether, of electric discharges and waves, of luminescence, and of radiation from hot bodies. Stefan's law, coefficients of absorption and emission, Kirchoff's law, Wien's law, Planck's formula, and the pressure of light are then treated. In dealing with the velocity of light the difference between wave velocity and group velocity is mentioned, as it should be. There is then a transition to topics in geometrical optics, reflexion, refraction, and the theory of mirrors and lenses. The facts about the index of refraction, including the case of metals, are presented with reference to various formulas which are stated without derivation. The subject is logically continued into dispersion and a lengthy discussion of spectrum analysis. There is a note by de Gramont on modern methods of spectrum observation.

After treating the transformation of radiant energy (fluorescence, phosphorescence, photography) and the measure of radiant energy (photometry, actinometry), the author goes over to optical instruments (microscope, telescope, and the like), the elements of physiological optics, and atmospheric optics. Not until then does he come to interference, and later to diffraction. These subjects, however, are covered in detail in some 120 pages. There is nothing of mathematical theory except a short note by the translator on retarded potentials and their application to problems in diffraction. In setting forth the phenomena connected with polarization the usual rather numerous trigonometric formulas are given without any complicated mathematical analysis. The translator, however, adds a note on Green's theory of reflexion and refraction based on the notion of an elastic medium and following Lord Kelvin's presentation in his Baltimore lectures. In this connection there is a short section on the gyrostatic theory of light.

The various sorts of elastic solid theory of light which have been developed from Green to Kelvin, have undoubtedly had very eminent adherents, have still a prominent place in optical theories of the more mathematical kind, and have certainly exercised a great influence over the development of optics. Their historical prominence is therefore such that they cannot be ignored. But it is difficult to repress the feeling that they have little place in the present work. And this for at least two reasons. First they were pretty thoroughly demolished by Gibbs's comparison of Kelvin's work with the electromagnetic theory. Second they do not accord with the strictly modern, maxwellian, point of view of the author. It may be that this discord will be remedied in the fourth volume. We should not have mentioned the matter were it not for the rather elaborate extension of old fashioned theories in the following treatment of crystalline optics. It may not be amiss further to state that from an elementary, though rather abstract, point of view it would have been better at this point to have founded the theory of light upon Gibbs's almost kinematic discussion of a wave motion in a fine grained medium. This too would have been somewhat nearer to the method of Boussinesq who, though generally ranked among the elastic solid theorists, is really very careful to indicate the ways in which the ether fails to act as a solid and to avail himself of general physical reasoning more or less independent of elastic theories.

The author's final three chapters on double refraction, interference of polarized light, and rotation of the plane of polarization, covering about 200 pages, are like most of the volume chiefly descriptive. Such subjects as accidental anisotropy and liquid crystals are not forgotten or slighted. The wonder is how Chwolson could so thoroughly and acceptably have collected and arranged and set forth such a vast variety of physical phenomena and theories as appear in the first half of his work. We must reserve the second half for notice at another time.

The volume should close here, but does not. For there is a note of 225 pages of the most mathematical character by the two Cosserats on the theory of deformable bodies. Even more than their note in the first volume is this wholly out of place, a worse than vermiform appendix. If the substance of these notes were unimportant, we could ignore them; but as they contain material which forms a distinct contribution to theoretical analytic mechanics, we shall try to give our readers an idea of their content at an early date.

From the numerous adverse comments which we have passed upon Chwolson's two volumes it might appear that we were not pleased with them. This is not so. We have found all the fault we could (a great deal of it with the editors' additions) in the hope that in future editions our criticisms might perhaps be productive of improvements. But without any changes the work is a masterpiece, and one that we should be glad to see not only in physical libraries but upon the shelves of our mathematical seminar rooms, to the end that our students of mathematics might readily avail themselves of its interesting and readable and broadening contents.

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ARITHMÉTIQUE GÉNÉRALE.

Je trouve dans le BULLETIN (Mars, 1912) un compte-rendu de mon livre Arithmétique générale.

Je crois nécessaire et de mon droit de rectifier une grave erreur du critique. Mon arithmétique n'est pas du tout un corollaire de la physique. J'ai dit au contraire que la mathématique pure doit être à mon avis une préface de la physique. Cela veut-il dire que la mathématique doive être une science expérimentale? Nullement. Or le critique donne à entendre que ce soit là ma façon de penser. Il dit que je démontre mes théorèmes fondamentaux en me servant de "remarques faites expérimentalement sur la nature des grandeurs" et le lecteur de ce compte-rendu doit s'imaginer

que ces grandeurs sont des grandeurs physiques, ou naturelles. Cela est absolument faux.

Le critique a lu ma préface, mais il n'a pas lu le livre. Sans quoi il n'aurait pas traité ma théorie de "naïveté." Je suis aussi rigoristé que n'importe qui, et je serais le premier a traîter, non de naif, mais de farceur, celui qui prétendrait baser la mathématique, science de raison, directement sur la physique, science expérimentale.

En réalité j'ai pris pour base de mon Arithmétique générale, le rapport de deux grandeurs géométriques de même espèce: deux segments de droite, deux angles, deux fuseaux d'une même sphère, deux parallelépipèdes rectangles de même base, etc. (voir à ce sujet, ce que j'ai dit au No. 3, page 2).

J'ai reconnu, depuis la publication de mon livre, que je n'ai fait que reprendre d'une façon plus complète la base posée par Duhamel.* Celui-ci appelle rapport de deux grandeurs, leur manière d'être relative. C'est exactement la même idée que celle que j'ai développée. Mon arithmétique est donc basée sur la géométrie euclidienne, et non sur la physique. Il est vraiment étonnant que ce qu'on appelle "l'admirable théorie des proportions d'Euclide, la précision et la netteté remarquables de Duhamel, et l'admirable théorie des quaternions de Hamilton, deviennent des naïvetés quand elles sont exposées par Dumont. (Voir à ce sujet, l'Encyclopédie des sciences mathématiques, tome I, volume I, fascicule 1, pages 134 et 135).

M. Lennes n'a pas lu mon livre. Il est resté hypnotisé devant une réflexion que j'ai émise dans ma préface à propos de la définition d'une somme géométrique de vecteurs. J'ai dit que cette définition n'a pas été choisie au hasard, ni afin de satisfaire au principe de permanence de Hankel, mais bien afin de pouvoir être utilisée en mécanique pour la composition des forces, des vitesses, etc.

Cette réflexion m'avait paru utile à dire, parce que j'avais lu peu auparavant un ouvrage de M. Dassen: "Des quantités mathématiques" où cet auteur explique que si l'on a choisi la définition de la somme géométrique des vecteurs telle que chacun la connait, c'est afin de pouvoir intervèrtir les termes, comme dans toute somme: $\overline{AB} + \overline{CD} = \overline{CD} + \overline{AB}$. Eh bien, en voilà une de naïveté! Car si l'on veut jeter un coup d'oeil sur la théorie des quaternions, on y rencontrera une somme géométrique d'angles (ou d'arcs de grand cercle sur une sphère), somme qui n'est pas commutative, et qui est l'angle d'un produit de quaternions.

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* Des méthodes dans les sciences de raisonnement, pp. 62–75.

Et c'est cette malheureuse réflexion où j'ai parlé de forces, qui a faussé toute la compréhension que M. Lennes s'est faite de ma théorie.

Il n'a pas vu que j'ai tout simplement adapté à tous les nombres, nombres absolus, nombres qualifiés et nombres complexes, les définitions qu'Hamilton avait créées pour les quaternions.

Lorsque j'ai parlé d'expérience, j'ai également été mal compris. Il ne s'agit d'expériences à faire sur des grandeurs physiques, il s'agit de l'expérience, de la connaissance que le penseur a du monde qui l'environne. Or la géométrie est précisément l'image idéale de ce monde. L'homme l'a créée afin de pouvoir plus facilement étudier l'univers. Et c'est donc la géométrie, science rigoureuse, que j'ai prise comme base essentielle de mon Arithmétique, afin que l'arithmétique concours immédiatement, comme la géométrie, à l'étude de l'univers.

Les théories arithmétiques des logiciens ne peuvent satisfaire un esprit hanté par le problème de la rotation de la terre dans le champ magnétique du soleil. Ces théories elles-mêmes qui se piquent d'être pures de tout compromis avec les notions expérimentales (physiques) poursuivent une utopie et se bercent d'illusion. Voici en quoi. Elles se bercent d'illusion en ce sens que toutes ces théories reposent sur la notion d'ordre. La première définition, pierre fondamentale de tout l'édifice, est la suivante: On appelle nombres entiers, une suite de mots ou de signes se succédant dans un ordre caractéristique, arbitrairement choisi une fois pour toutes. Ces signes n'ont aucune signification par eux-mêmes. Ainsi 5 est le signe qui suit 4 et qui précède 6. Rien de plus. La suite commence au caractère 1:

$$1.\ 2.\ 3.\ 4.\ 5.\ 6.\ 7.\ \cdots.$$

Eh bien, dire que 5 vient après 4, ou suit 4, ou succède à 4, ou est à droite de 4, c'est faire appel à la notion temps! ou à la position des organes dans le corps humain (la droite et la gauche).

Les autres définitions, fraction, nombre irrationnel, nombre qualifié, nombre complexe, c'est-à-dire, tous groupes de deux nombres écrits dans un ordre caractéristique, font également appel à cette notion d'ordre, c'est à dire, de temps. C'est, je pense, ce qui avait fait dire à Gauss que l'arithmétique est la science du pur temps.

C'est enfin une utopie que de prétendre émettre des pensées débarrassées des concepts fondamentaux que le monde où nous vivons a mis dans notre conscience; pensées qui n'en sont que le produit, synthétisé grâce à notre mémoire consciente ou non.

Pour en revenir à ma méthode, je dirai que je suis aussi désireux que n'importe qui, de découvrir où se cachent des trous dans mes démonstrations, et le cas échéant, comment il faudrait les combler. Malheureusement les critiques n'ont pas le temps de lire consciencieusement les livres qu'ils doivent analyser. Il en est en mathématique comme dans les salons bien fréquentés: Chacun ne s'intéresse à la conversation que pour la part qu'il y prend.

J'espère que vous voudrez bien insérer cette réponse à votre compte-rendu, et que M. Lennes reconnaîtra de bonne foi qu'il c'est trompé quant à la valeur scientifique de mon point de départ.

E. DUMONT.

I do not deem it necessary to refer in detail to the strictures which Professor Dumont has made on my recent review of his Arithmétique générale. I need only to repeat that there are no explicitly stated postulates in the book, that what are usually regarded as such in arithmetic are proved as theorems by general reference to statements about magnitudes in general. These statements about magnitudes in general are obtained quasi-inductively from geometric considerations. In this sense the book is truly naïve, and in this sense Hamilton's treatment may well be regarded as naïve.

N. J. LENNES.

SHORTER NOTICES.

The Theory of Determinants in the Historical Order of Development. Volume II: *The period* 1841 *to* 1860. By THOMAS MUIR, Superintendent-General of Education in Cape Colony. London, Macmillan and Company, 1911. xvi + 475 pp.

THE general arrangement of the present volume is similar to that of volume I, which appeared in 1906 and was reviewed in the BULLETIN, volume 13 (1907), page 244. It is composed, mainly, of a list of books and articles which appeared during the twenty years from 1841 to 1860, and of brief remarks on their contents. A chapter is assigned to each of the following thirteen special forms: axisymmetric determinants, alternants, compound determinants, recurrents, Wronskians, Jacobians, skew determinants, orthogonants, persymmetric determinants, bigradients, Hessians, circulants, and continuants. The last chapter deals with less common forms,—including permanents, determinants with complex elements, determinants connected with anharmonic ratios, and Sylvester's unisignant.

The first two chapters of the book are devoted to determinants in general, and cover 108 pages. In the chapter on continuants the author has gone beyond the limits to which he confined himself in the other parts, by pursuing this subject up to 1870. Articles and books which relate to different forms are considered under each of the forms with which they have direct contact, and hence the same work is often listed and considered in a number of different chapters. The book closes with a list of the 87 different authors "whose writings are reported on," but there is no subject index. Such an index would have been especially useful in view of the large number of different terms employed by the various authors whose writings are considered.

It was more than thirty years ago that Muir began the collection of the material which has served as a basis for the present work, and his extensive lists of writings on determinants, which have appeared in the *Quarterly Journal of Mathematics,* volumes 18, 21, and 36, are well known. Hence one does not expect to find many omissions in a work relating to the comparatively short period considered in the present volume. It seems, however, that a history of determinants during this period should not fail to mention Weierstrass—especially his article which appeared in 1858 in the Berliner *Berichte.*[*] In the history of Hermitian determinants (page 449) reference should also have been made to the articles by Hermite which appeared in volume 47, page 345, and in volume 53, page 183, of *Crelle.*

What appears to the reviewer as the most serious defect in the present work results from the fact that the chronological arrangement of the articles is frequently based solely on the dates printed at the end of the articles, without any reference to the date of publication. For instance, Spottiswoode's memoir entitled "Elementary theorems relating to determinants, second edition," which appeared in *Crelle* in 1856, is classed with the papers which appeared in 1853. The unreliability of the dates at the end of articles was discussed by Eneström in the *Bibliotheca Mathematica,* volume 5 (1904), page 196; and, in this instance, Eneström seems to agree with Cantor's dictum, "da die Geschichte unwiderruflich die Veröffentlichungszeit als allein massgebend betrachten muss, wo Erstlingsrechte zu vergeben sind . . ."[†] It seems to the reviewer that there is a growing tendency to omit dates at the end of articles and he believes that this is desirable.

Among the few minor errors which the reviewer has noticed is one on page 206 relating to the date of an article by Bazin, entitled "Sur une question relative aux déterminants," which appeared in *Liouville,* volume 16 (1851). This article is classed in the present volume with those of 1854 and the date assigned to it is April, 1854. The number of minor errors seems, however, small; and, while the reviewer does not consider the present volume to be as complete and useful as volume I, yet it must be regarded as an important addition to the literature on determinants, and it involves a large amount of interesting information in a fairly convenient form. The fact that its author is continuing his mathematical work notwithstanding his administrative duties is an element of inspiration.

G. A. MILLER.

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[*] Cf. Pascal's Repertorium, vol. 1 (1910), p. 66.

[†] Cantor, Vorlesungen über Geschichte der Mathematik, second edition, vol. 2 (1900), p. 811.

An Introduction to the Lie Theory of One-Parameter Groups with Applications to the Solution of Differential Equations. By Dr. ABRAHAM COHEN. D. C. Heath and Company, 1911. vii + 248 pp.

THE theory of continuous groups has become a subject of such general importance, and constitutes a method of such great power, that every attempt to render it more easily accessible should be received gratefully. One of the advantages of the Lie theory is its elementary character, in so far as a large portion of it may be developed with very modest prerequisites. So much the more must we welcome a book which enables us to present this subject to undergraduate classes in our own language. It is always a serious thing to complicate matters for the younger students by adding linguistic to their inevitable mathematical difficulties. The large works of Engel and Scheffers on Lie's theory are written with admirable pedagogic insight, so that there would be little need of such a book as this in German. But the only larger English book on the subject, that by Campbell, is far from being a paragon of lucid exposition. The little book by Page is hardly extensive enough to serve the purpose of more than a mere introduction.

There is then a real field for such a book as Dr. Cohen's. We feel, however, as though the author had overemphasized the differential equation part of the theory. Still, that is a matter of taste.

The book is divided into seven chapters and an appendix. The first chapter naturally begins with the definition of a group of transformations, infinitesimal transformations, invariants, path curves, canonical variables, etc. In Chapter II these notions are applied to differential equations of the first order in connection with the theory of the integrating factor. The third chapter has the heading Miscellaneous theorems and geometrical applications, and deals among other things with isothermal curves. Chapter IV is on differential equations of the second and higher orders, and introduces the notion of n-times extended group. Chapter V is devoted to linear partial differential equations of the first order, while Chapter VI returns to ordinary differential equations of the second order. Chapter VII finally is on contact transformations. In the appendix are developed a number of the more difficult parts of the general theory, these matters being referred to in the proper place in the main body of the text. Besides, the appendix contains two tables to serve as an aid in properly classifying and integrating differential equations.

The book is well written and is well adapted as a general introduction to Lie's theory. As already indicated, the too systematic treatment of differential equations from this point of view appears to the reviewer to be a defect. It is one thing to recognize that Lie's theory is capable of bringing some order into the haphazard methods of the elementary theory of differential equations; but is is quite another to use Lie's theory as a strait jacket which every differential equation must be made to fit.

And now for a few points of detail. The author's definition of a one-parameter group is rather clumsy, as a result of the unfortunately so common desire to put it all into one sentence. The two parts of remark 1 on page 7 are not obviously connected. The true reason for the validity of remark 1 is to be found in the latter part of remark 2. Although the author speaks everywhere of one-parameter groups, he really means such groups as are generated by an infinitesimal transformation. At least, a number of his theorems are not true for mixed groups. On page 176 in deriving the condition for a "union of lineal elements" he neglects the case where the union consists of the elements of a fixed point. The printing is well done. The reviewer has noticed only a few misprints. On page 45 last line read $-A\eta$ instead of $+A\eta$. Page 177 in Ex. 5 read $-\cot t$ instead of $-\cos t$. Page 188 just above equation (130′) read characteristic instead of charactertisic.

A rather confusing feature is the use of two different scarcely distinguishable fonts for the same letters, as on page 24. There is no necessity for this. It seems a pity that the publishers have not seen fit to give the book a wider margin. The binding also is none too good.

E. J. Wilczynski.

Abriss der Algebra der Logik. Bearbeitet im Auftrag der Deutschen Mathematiker-Vereinigung von Dr. Eugen Müller, Professor an der Oberrealschule zu Konstanz. In drei Teilen. Erster und Zweiter Teil. Leipzig und Berlin, B. G. Teubner, 1909–1910, 159 pp. 8 marks.

Almost from the days of Boole an ever-increasing need has been felt for an elementary introduction to the growing subject of the algebra of logic. There have been numerous extended and more or less exhaustive treatises on the subject, notably that of Dr. Ernst Schröder, but few of an elementary character.* The present work has the name of Dr. Schröder at the top of title page and was inspired by the references in several places in Schröder's Vorlesung to a proposed "Abriss." The first part, which serves as an introduction, was partly prepared from an outline and manuscript fragments left by Schröder. The first twenty pages are given up to the elementary introduction to the ideas, terms, and symbols used. The standpoint used here is that of the algebra of propositions or implications. This part of the book is rather diffuse and the entire work contains much that is obvious, and could be condensed with advantage. This is probably intended to appeal to the non-mathematical or philosophical reader. The seven axioms on which the subject is based are given on pages twenty and twenty-one. This part is interpreted from the standpoint of the algebra of inclusion or classes or of incident regions, and the two phases are treated from this point on in parallel.

The axioms are as follows:

I. $a ⋹ a$, Identitäts- oder Tautologieaxiom.
II. $(a ⋹ b)(b ⋹ c) ⋹ (a ⋹ c)$, Subsumtionsschluss.
III. $(a ⋹ b)(b ⋹ a) = (a = b)$, Gleichheitsdefinition.
IV×. $0 ⋹ a$, Nullpostulat. IV+. $a ⋹ 1$, Einspostulat.
V. $1 \not⋹ 0$ Existenzpostulat.
VI×. $(x ⋹ a)(x ⋹ b) = (x ⋹ ab)$, Produktdefinition.
VI+. $(a ⋹ y)(b ⋹ y) = (a + b ⋹ y)$, Summendefinition.
VII × +. $(a + z)(\bar{a} + z) = z = az + \bar{a}z$, Negations- oder Distributionsprinzip.

These axioms are explained and some alternate axioms are given in the next five pages. The rest of part one (to page 50) is taken up with the deduction of theorems which are the immediate consequences of these axioms. Most of the relations in part one are expressed in terms of the asymetrical sign of implication or inclusion under the general term of "subsumption" (⋹), and the sign of equality is defined by Axiom III in terms of the sign of subsumption. There is a logical fallacy involved here in using the equality sign in its own definition.* A better definition would be composed of the two parts†

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* Except the work by Couturat I know of no other of a general character.

$$(a = b) \mathrel{-\!\!\!\!<} (a \mathrel{-\!\!\!\!<} b)(b \mathrel{-\!\!\!\!<} a), \quad \text{and} \quad (a \mathrel{-\!\!\!\!<} b)(b \mathrel{-\!\!\!\!<} a) \mathrel{-\!\!\!\!<} (a = b).$$

The equality sign is used however before Axiom III is given. In part two, Aussagentheorie, Funktionen, Gleichen und Ungleichen, the author derives and discusses at length the general formulas for the symbolic sum and product, and the methods of substitutions and reduction of symbolical functions with the methods of elimination and solution of the general symbolical equation. This part is quite extensive and there is much that is obvious.

One good feature of the work, at least from the standpoint of the mathematician, is the numerical or algebraical examples, illustrative of the general theory, used throughout the book. The Abriss on the whole is a very creditable piece of work and cannot but help arouse interest in the algebra of logic. No serious errors occur, but there are some obvious misprints of the type found in line three, page thirteen, where a in the last parenthesis should be α.

L. I. NEIKIRK.

Cours d'Astronomie. Première partie: *Astronomie théorique.* Par H. ANDOYER. Deuxième édition entièrement refondue. Paris, Librairie scientifique A. Hermann. 383 pp.

THE first edition of this work in hectographed form has been reviewed in volume 13, number 10, of the BULLETIN.‡ The second edition is much enlarged, almost doubled in size. The number of chapters is increased from fifteen to nineteen. The work in its present form is subdivided into four large sections, or books, of which the first is of a more analytical character. The transformation of coordinates with common origin—spherical trigonometry—is given with considerable completeness; to it is added the transformation for different origins and parallel systems of axes—the analytical basis for parallax and aberration. The second book defines the adopted systems of coordinates in astronomy, the concepts of sidereal and solar time, and gives, as an application of the theories of the first book, the theories of parallax and aberration themselves. This book contains likewise some fundamental notions about geodesy, which is in proper bearing with the greater extent with which the theory of spherical trigonometry has been given in this work. That the theory of refraction is likewise included in this second section is of course due to the fact that the correction for refraction comes, in the order of application, even before those for parallax and aberration. The third book opens with a rather complete study of certain fundamental notions of celestial mechanics which are made use of in the theory of precession and nutation and of time, all of which theories are set forth in this section. The geocentric motions of planets and satellites, including the moon, complete the third book; their inclusion, in the text before us, is much to be commended, since the older textbooks on this subject give almost no attention to them. Inasmuch as observations of eclipses of moon and sun and occultations of stars define certain directions in space, to which the usual corrections are to be applied, the last section of the work is devoted to their theories. Moreover the theory of transformation of coordinates finds in the prediction of the times of eclipses an extensive field of application.

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* A similar fallacy is evident in axiom VI×. It would be difficult to avoid this in a simple way.

† A combination of his III′, III″ and III‴.

‡ See also the review of the second part, by Professor Longley, BULLETIN, vol. 15, pp. 467–468.

The name of the author and the reputation of the publishing house are from the start sufficient guarantees for the excellent character of the book. As far as has been possible to ascertain, typographical errors of any importance seem to be happily avoided; that the types for the fractional lines have not always been perfect is perhaps the only criticism one can make of the typographical appearance of the book. The introduction of a general index and copious references at the end of chapters would have been a distinctive improvement; as it stands, the beginner is likely to miss the historical setting, and permit himself to be carried too much by the "ipse dixit" of the author. It is believed that the mathematician, interested in the subject, will find Professor Andoyer's book extremely attractive, since everywhere the underlying mathematical principles are strongly brought into the foregound. His attention is especially called to the chapter on refraction, in which Radau's presentation is given in preference to those of Laplace, Bessel, or Ivory. A simple reference to Bruhns's Atmosphärische Strahlenbrechung would have supplied the reader with a pretty complete literature of this highly interesting subject.

The reviewer would have liked to see a chapter on interpolation and related subjects form part of the book, since an intelligent use of the Ephemerides is not to be thought of without a mastery of this topic. The astrophysicist will be somewhat disappointed by not seeing the transformation of velocities included in the third chapter of the book. The student interested in the photometric measurements of the satellites of Jupiter after Pickering and Sampson might well claim that the theory of the photometric eclipses is surely as important as the passages of Mercury and Venus over the disc of the sun, to which the author has devoted 8 pages. Finally in the subject of differential refraction a recognition of the peculiar formulas for measurement of photographic plates might not have been entirely out of place.

KURT LAVES.

NOTES.

AT the meeting of the Edinburgh mathematical society on May 10 the following papers were read: By W. P. Milne, "Investigations on circular cubics and bicircular quartics"; by G. Teixeira, "Notice sur les recherches de Maclaurin concernant les cubiques circulaires"; by H. T. Pioggio, "Note on linear differential equations with constant coefficients."

THE annual meeting of the Deutsche Mathematiker-Vereinigung will be held in affiliation with the eighty-fourth convention of German naturalists and physicians at Münster in Westphalia September 15–21 under the presidency of Professor W. v. DYCK. While papers in other subjects will be welcome, particular emphasis will be put upon reports and new contributions to the theory of differential geometry. Information regarding the meeting can be obtained from the secretary, Professor A. Krazer, Karlsruhe, Westendstrasse 57.

THE Macmillan Company announce the publication during the summer of a work on the calculus by Professors E. W. DAVIS and W. C. BRENKE, of the University of Nebraska.

THE second volume of Professor J. PIERPONT'S Theory of functions of a real variable is in press and will be published by Ginn and Company in July.

HENRY Holt and Company announce as in preparation a work on analytic geometry of space by Professors Virgil SNYDER and C. H. SISAM.

UNIVERSITY OF MICHIGAN. The following advanced courses in mathematics are announced for the academic year 1912–1913: By Professor W. W. BEMAN: Advanced calculus, two hours; Higher plane curves, two hours, second half-year; Linear differential equations, two hours, second half-year.—By Professor ALEXANDER ZIWET: Theory of potential, three hours, first half-year; Hydrodynamics, two hours; Vector analysis, three hours, second half-year; Integral equations, two hours, second half-year.—By Professor J. L. MARKLEY: Theory of functions I, three hours; Theory of functions II, two hours.—By Professor J. W. GLOVER: Seminary in probabilities, two hours.—By Professor W. B. FORD: Harmonic analysis, two hours; Infinite series and products I, two hours; Infinite series and products II, two hours.—By Professor PETER FIELD: Theory of twisted curves and surfaces, two hours; Advanced mechanics, three hours.

PROFESSOR J. HADAMARD has been appointed professor of mathematical analysis at the Ecole polytechnique of Paris, as successor to Professor CAMILLE JORDAN, retired.

THE Göttingen academy has awarded M5,000 from the Wolfskehl foundation to Professor E. ZERMELO, of the University of Zurich, for his researches in the theory of sets.

DR. L. SCHRUTKA, of the technical school of Vienna, has been appointed associate professor of mathematics at the German technical school at Brünn.

THE University of Colorado has conferred the honorary degree of doctor of laws on Professor FLORIAN CAJORI, of Colorado College.

PROFESSOR VIRGIL SNYDER, of Cornell University, has been granted leave of absence during the first half of the academic year 1912–1913, which he will spend in travel abroad.

AT the University of Minnesota, Professor H. T. EDDY, head of the department of mathematics and mechanics of the college of engineering and dean of the graduate school, has retired from active service. Professor W. E. Brooke has been appointed to succeed Professor Eddy. Dr. W. F. Holman has been appointed to a professorship of engineering mathematics, to succeed Professor A. E. Haynes, retired.

Catalogues of second-hand books: Henry Sotheran and Co., 140 Strand, London, catalogue 725, mathematics and allied subjects, about 2,000 titles.—G. E. Stechert and Co., 151 West 25th Street, New York, scientific journals and proceedings, about 700 titles.—Emile Blanchard, 10 rue de la Sorbonne, Paris, catalogue 65, 1,167 titles in pure and applied mathematics.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Bergholz (O. A.). Die Losung des Fermatschen Problems $x^n + y^n = z^n$. Dessau, Art'l, 1912. 8vo. 19 pp. M. 1.00

Blenck (G.). Untersuchungen über das Amiotsche Theorem bei den Flachen zweiter Ordnung und uber Erzeugungsarten des elliptischen Kegels. (Diss.) Rostock, 1911. 8vo. 91 pp.

Bluhm (B.). Ueber konjugierte Kurven und Flächen. (Diss.) Königsberg, 1911. 8vo. 89 pp.

Böger (R.). Symmetrische Involutionen und fokal getrennte Geraden. (Progr.) Hamburg, 1912. 8vo. 56 pp.

Brincard (R.). See Duporcq (E.).

Burali-Forti (C.). Corso di geometria analitico-proiettiva. Torino, Gallizio, 1912. 8vo. 7+268 pp. Cloth. L. 8.00

Carslaw (H. S.). An introduction to the infinitesimal calculus. 2nd edition. London, Longmans. 8vo. 4s.

Chemnitzer (B.). Die kürzesten Linien auf der Kreisringfläche. 1ter Teil. (Progr.) Chemnitz, 1912. 4to. 19 pp.

Crefcoeur (A. J. M.). Cours d'analyse infinitésimale. 3e partie: Calcul des probabilités ou théorie analytique du hasard. 2e édition, revue et augmentée. Liège, Béranger, 1911. 8vo. 253 pp. F. 7.00

Dedekind (R.). Stetigkeit und irrationale Zahlen. 4te, unveränderte Auflage. Braunschweig, Vieweg, 1912. 8vo. 7+24 pp. M. 1.00

——. Was sind und was sollen die Zahlen? 3te, unveranderte Auflage. Braunschweig, Vieweg, 1912. 8vo. 21+58 pp. M. 1.80

Duporcq (E.). Premiers principes de géométrie moderne. 2e édition, revue et augmentée par R. Brincard. Paris, Gauthier-Villars, 1912. 8vo. 7+175 pp.

Eckelhardt (F.). Differential- und Integralrechnung. Wien, Pichler, 1912. M. 0.85

Encyclopédie des sciences mathématiques pures et appliquées. Edition française. Tome I. 2e volume: Algèbre. Fascicule 4: Théorie des formes et des invariants, par F. W. Meyer et J. Drach. Leipzig, Teubner, 1912. 8vo. Pp. 425–520. M. 3.60

Festschrift. Heinrich Weber zu seinem 70. Geburtstag am 5. März, 1912. Gewidmet von Freunden und Schülern. Mit einem Bildnis von H. Weber. Leipzig, Teubner, 1912. M. 24.00

Funk (P.). Ueber Flächen mit lauter geschlossenen geodätischen Linien. (Diss.) Göttingen, 1911. 8vo. 22 pp.

Gyr (W. X.). Die Polaren der Lemniskate. (Diss.) Bern, 1911. 8vo. 59 pp.

Heffter (L.). Ueber eine vierdimensionale Welt. (Öffentliche Antrittsrede.) Freiburg, Speyer und Kaerner, 1912. 8vo. 26 pp. M. 0.90

Hermite (C.). Oeuvres, publiées sous les auspices de l'Académie des sciences, par E. Picard. Tome III. Paris, Gauthier-Villars, 1912. 8vo. 528 pp. Fr. 18.00

Kneser (A.). Mathematik und Natur. (Rektoratsrede.) Breslau, Trewendt und Granier, 1912. 8vo. 14 pp. M. 0.50

LeMessurier (T. A.). Key to W. W. Johnson's Differential Equations. New York, Wiley, 1912. 8vo. 5+125 pp. $1.75

MacNeish (H. F.). Linear polars of the k-hedron in n-space. (Diss.) Chicago, University of Chicago, 1912. 8vo. 32 pp. $0.25

Neumann (R.). Die Entwicklung willkürlicher Funktionen nach den Hermiteschen und Laguerreschen Orthogonalfunktionen auf Grund der Theorie der Integralgleichungen. (Diss.) Breslau, 1912. 8vo. 111 pp.

Notarbartolo (G.). Metodo rapido con semplici addizioni per la costruzione di tavole delle potenze di numeri della serie naturale. Acqui, Tirelli, 1912. 8vo. 14 pp.

——. Valore costante delle differenze dell' n° ordine, ecc. Torino, Unione tipografico-editrice, 1912. 4to. 10 pp.

Picard (E.). See Hermite (C.).

Reider (R.). Polynomische Entwicklungen von Funktionen einer komplexen Variabeln. (Diss.) Basel, 1911. 8vo. 85 pp.

Rose (M.). Einleitung in die Funktionentheorie (Theorie der komplexen Zahlenreihen). (Sammlung Göschen, Nr. 581.) Leipzig, Göschen, 1912. 8vo. 139 pp. Cloth. M. 0.80

Sansone (G.). Sulle divisioni regolari dello spazio iperbolico in poliedri regolari e in tetraedri. (Tesi di laurea.) Pisa, Nistri, 1911. 8vo. 76 pp.

Steinhaus (H.). Neue Anwendungen des Dirichlet'schen Prinzips. (Diss.) Göttingen, 1911. 8vo. 45 pp.

Veneroni (E.). Sui sistemi di fasci di rette che costituiscono complessi cubici. Pavia, Bruni, 1912. 8vo. 16 pp.

Weber (H.). See Festschrift.

Wiener (F. W.). Elementare Beiträge zur neueren Funktionentheorie. (Diss.) Göttingen, 1911. 8vo. 40 pp.

Wilson (E. B.). Advanced calculus. A text upon select parts of differential calculus, differential equations, integral calculus, theory of functions. Complete edition. Boston, Ginn, 1912. 8vo. 9+566 pp. $5 00

Zechstein (P. G.). Die Regel des Pythagoras und die Behauptung des Fermat. Berlin, Buschhardt, 1912. 8vo. 4 pp. M. 2.00

II. ELEMENTARY MATHEMATICS.

Bates (E. L.) and Charlesworth (F.). Practical mathematics and geometry. Part 2: Advanced course. New York, Van Nostrand, 1912. 12mo. 776 pp. $1.50

Bôcher (M.). Graduate work in mathematics in universities and in other institutions of like grade in the United States. (International commission on the teaching of mathematics.) Washington, 1911.

Bourlet (C.). Algèbre. 1re, 2e et 3e années. Paris, Hachette, 1912. 16mo. 228 pp. Cartonné. Fr. 2.00

Bürklen (O. T.). Aufgabensammlung zur analytischen Geometrie der Ebene. 2te, verbesserte Auflage. (Sammlung Goschen, Nr. 256.) Leipzig, Göschen, 1912. 8vo. 175 pp. Cloth. M. 0.80

Charlesworth (F.). See Bates (E. L.).

Commissaire (H.). Leçons d'algèbre. Classes de 2e C et D. Paris, Masson, 1912. 8vo. 8+270 pp. Fr. 3.00

Crosara (L.). Trattato di geometria piana. 2a edizione. Torino, Paravia, 1911. 8vo. 7+155 pp. L. 2.00

Dalle (A.). 2000 théorèmes et problèmes de géométrie avec solutions. Namur, Wesmael-Charlier, 1912. 8vo. 8+825 pp. Fr. 15.00

—— et De Waele (C.). Cours de géométrie. Geometrie plane. 3e édition. Namur, Wesmael-Charlier, 1912. 8vo. 328 pp. Fr. 2.75

Faifofer (A.). Elementi di geometria. 17a edizione. Venezia, Sorteni e Vidotti, 1911. 8vo. 533 pp. L. 4.00

—— Il libro di geometria. 4a edizione. Venezia, Sorteni e Vidotti, 1911. 8vo. 153 pp. L. 1.50

Franchis (M. De). Complementi di geometria. Palermo, Sandron, 1911. 8vo. 215 pp. L. 3.00

Frink (F. G.). See Hall (A. G.).

Hall (A. G.) and Frink (F. G.). Trigonometric and logarithmic tables. New York, Holt, 1911. 8vo. 3+97 pp. $0.75

Hall (H. S.). A school algebra. Parts 2 and 3. With answers. London, Macmillan. 8vo. 2s. 6d.

——. A school algebra. Part 3. With or without answers. London, Macmillan. 8vo. 1s. 6d.

Hočevar (F.). Lehr- und Uebungsbuch der Geometrie. 7te Auflage. Wien, Tempsky, 1912. 8vo. 170 pp. Cloth. M. 2.80

Koschemann (O.) und Otten (K.). Lehr- und Uebungsbuch fur den mathematischen Unterricht an Mittelschulen. 3ter Teil: Geometrie. Frankfurt, Diesterweg, 1912. 8vo. 262 pp. Cloth. M. 2.40

Kutnewsky (M.). See Müller (H.).

Lietzmann (W.) und Schimmack (R.). Berichte, veranlasst durch die Internationale Unterrichtskommission. Heft 7: Der Kongress in Mailand vom 18. bis 20. September, 1911, von W. Lietzmann. Ueber die Verschmelzung verschiedener Zweige des mathematischen Unterrichts. Leipzig, Teubner, 1912. M. 1.60

Mehler (F. G.). Elementarmathematik. Bearbeitet von A. Schulte-Tigges. Ausgabe B. 1ter Teil: Synthetische Geometrie der Kegelschnitte. 2te, unveränderte Auflage. Berlin, Reimer, 1912. 8vo. 9+72 pp. M. 1.50

——. Elementarmathematik. Ausgabe B. 2ter Teil: Arithmetik, niedere Analysis, Trigonometrie and Stereometrie. 2te, unveränderte Auflage. Berlin, Reimer, 1912. 8vo. 8+169 pp. M. 1.50

Moss (C. E.) and Welpton (W. B.). Mathematics and science. (University Tutorial Series.) London, Clive. 8vo. 144 pp. Sewed. 1s. 6d.

Müller (H.). Die Mathematik auf den Gymnasien und Realschulen. 1ter Teil. 7te Auflage. Ausgabe B. Leipzig, Teubner, 1912. 8vo. 8+208 pp. Cloth. M. 2.20

—— und Kutnewsky (M.). Aufgaben aus der Arithmetik, Trigonometrie und Stereometrie. 2ter Teil. Ausgabe B. 4te Auflage. Leipzig, Teubner, 1912. 8vo. 12+320 pp. Cloth. M. 3.00

Otten (K.). See Koschemann (O.).

Pletschinger (I.). Rechenlösungen auf der Grundlage des logischen und funktionalen Denkens. Biberach, Dorn, 1912. 8vo. 8+232 pp. Cloth. M. 3.50

Röhr (H. A.). Schnellrechentafeln, enthaltend die Produkte von 2 × 1 bis 9 × 9999. Hamburg, 1912. 8vo. 35 pp. M. 1.60

Schimmack (R.). See Lietzmann (W.).

Schmehl (C.). Lehrbuch der Stereometrie. Giessen, Roth, 1912. 8vo. 11 pp. M. 0.60

Schubert (H.). Arithmetik und Algebra. 2te, durchgesehene Auflage. 5ter Neudruck. (Sammlung Göschen, Nr. 47.) Leipzig, Göschen, 1912. 8vo. 171 pp. Cloth. M. .800

Serrazanetti (G.). Elementi di algebra. Bologna, Zanichelli, 1912. 16mo. 100 pp. L. 1.25

Smolik und Heller. Raumlehre und darstellende Geometrie. Bearbeitet von K. Hahndel. 4te Auflage. Wien, Tempsky, 1912. 8vo. 211 pp. Cloth. M. 3.60

Suppantschitsch (R.). Lehrbuch der Geometrie, Trigonometrie und analytischen Geometrie. Wien, Tempsky, 1912. 8vo. 294 pp. Cloth. M. 4.40

Swanwick (F. T.). Elementary trigonometry. Cambridge, University Press, 1911. 8vo. 259 pp. Cloth. $1.25

Switzer (S. A.). Practical geometry for schools. London, Methuen. 8vo. 170 pp. 2n.

Waele (C. De). See Dalle (A.).

Welpton (W. B.). See Moss (C. E.).

Wenzel (G.). Lehrbuch der Arithmetik. Wien, Tempsky, 1912. 8vo. 302 pp. Cloth. M. 4.00

——. Lehrbuch der Geometrie. Wien, Tempsky, 1912. 8vo. 242 pp. Cloth. M. 3.50

III. APPLIED MATHEMATICS.

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